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SPACE TRAVEL OF CHARGED TEST PARTICLES HITTING THE SINGULARITY OF A CHARGED CENTRAL GRAVITATIONAL CENTER

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Abstract— **We revisit the Kaluza-Klein theory and solve the field equations of the Kaluza-Klein theory with constant coupling field between the electromagnetic and gravitational field in terms of power expansions in the   coordinate for the spherical symmetric case entirely, where   is the distance between the gravitational center and the test particle. In the Einstein-Maxwell case where the electromagnetic field and the gravitational fields are coupled linearly, we discuss the exact behavior of the roots of the pseudo potential for the motion of the position   as a function of the planar -angle for an orbiting particle. We investigate the analytic continuation of a trajectory of a test particle entering the gravitational center of a central body, which has performed a temporal jump when exiting the gravitational center again. This temporal displacement, if repeated, constitutes a stochastic process that has an expectation value of the reduced Planck constant divided by two times the rest mass of the electron, since the temporal displacement process of the electron goes along with an annihilation and recreation process of the electron that enters and exits the gravitational center. Thus, our finding corresponds to the existence of a Heisenberg uncertainty relation with respect to temporal and energetic fluctuations of the electron in the electron-proton system, which translates to an Heisenberg uncertainty relation with respect to spatial variations and variations in the momentum of the electron in the electron-proton system. The validity of the latter uncertainty relation is equivalent with the existence of a Schroedinger equation governing the statistic behavior of the electron in the electron-proton system. In this way we derive the ground principles of classical quantum mechanics from the unified gravitational theory for gravitation and electromagnetism straightforwardly.**

***Keywords*— Privacy, Preservation, community, detection, Anonymization, social network.**

I. INTRODUCTION

In 1921 Kaluza [1921] presented a generalization of Einstein's original general relativistic theory, where the four-dimensional mathematical concept of differential geometry for the description of a four-dimensional space-time underlying Einstein's theory was generalized to five dimensions to describe a five-dimensional space-time. In this description the fifth dimension of the five-dimensional space-time was filled with the four components of the vector potential of the electromagnetic field plus a scalar coupling field between the electromagnetic field and the gravitational fields. This implementation of the electromagnetic field in the five-dimensional metric was done in a way that the electromagnetic field transformed as a gauche-field under a coordinate transformation of the five-dimensional space-time. The field equations for the four-dimensional metric, the vector potential and the scalar field are then obtained by calculating the Euler-Lagrange derivatives of the five-dimensional curvature of the five-dimensional metric and assuming that all fields are only depending on the coordinates of the four-dimensional space-time. This procedure secures that the Lagrange-density of the entire physical system, in this case the curvature of the five-dimensional space-time, is minimal, thus indicating that electromagnetic fields generate gravitational fields or that gravitational field induce electromagnetic fields, whereby relaxations of that kind take place under the boundary condition that time-dilation effects and Lorentz-contraction effects compensate each other to the largest possible extend. This treatment of the transformation of electromagnetic and gravitational field into each other is analogous to the treatment of classical mechanical systems in which kinetic energy transforms to potential energy and vice versa. In fact, the Kaluza [1921] approach is the generalization of that classical principle.

The Lagrange-density of the five-dimensional space can be extended by another action term, the so-called Gauss-Bonnet action, another interaction term between the electromagnetic and gravitational fields that leads to field equations of second order, and which has been proven to be unique Lovelock [1971]. The field equations without consideration of the mathematically possible Gauss-Bonnet term are called the field equations of the Einstein-Maxwell theory, whereas the field equations with consideration of the Gauss-Bonnet term are called the field equations of the full Kaluza-Klein theory (Klein [1926] gave the whole theory a quantum-mechanical interpretation in 1926). In the full Kaluza-Klein theory the gravitational and electromagnetic fields are coupled non-linearly, whereas in the Einstein-Maxwell theory a linear relationship between gravitational and electromagnetic terms holds. If the scalar field is considered to be a spatially and temporally varying field, there is another field equation for the scalar field in the non-linearly coupled theory. The full set of the field equations has been derived as late as 2015 [see, e.g., Williams, 2015]. In the case that the coupling field is a near constant, the field equations are much simpler and were derived first by Mueller-Hoissen [1988]. Mueller-Hoissen & Sippel [1988] discussed properties of the solutions of the field equations in the spherical symmetric case.

Schmidt [1990] expanded the investigation of Mueller-Hoissen & Sippel [1988] and discussed and classified the numerical solutions of the geodetic equations of the Kaluza-Klein theory with constant coupling field in the spherical symmetric case. Schmidt [1990] also found exact solutions of the field equations of the Kaluza-Klein theory with constant coupling field in the cylindrical symmetric case for a charged wall and a charged staff.

Schmidt [1990] also revisited the Einstein-Maxwell theory and integrated the geodetic equations of that theory for a moving test particle without rotational momentum. Schmidt [1990] integrated the geodetic equations of the Einstein-Maxwell theory for a moving test particle with rotational momentum up to the extend that analytic formula for the trajectories as functions of the rotation angle in the plane of motion were obtained, too.

Although full solutions of the geodetic equations of the Einstein-Maxwell theory even for the more general cylindrical symmetric case exist [see, e.g., Carter, 1968; Misner, 1973, equations 33.37a-d], these solutions are complicated by the fact that they contain severe components of a latitudinal motion of the test particle, which can only be expressed with nested elliptic integrals, thus making these solutions difficult to assess. However, in the approach of Schmidt [1990], in the integration process, a continuous rotation of the coordinate system is introduced, which is a free parameter for any solution of general relativistic geodetic equations that shows that the solution in the spherically symmetric case is actually a movement of the particle around the gravitational center in a plane, the ecliptic, and that these solutions can be expressed in terms of simple elliptic functions. So the solutions of Schmidt [1990] have the quality of having obtained generalized Kepler ellipses for the Kepler problem within the Einstein-Maxwell theory.

In this work we revisit earlier work regarding the Kaluza-Klein theory, and expand the work of Mueller-Hoissen & Sippel [1988] in order to obtain the full solutions of the field equations of the Kaluza-Klein theory with constant coupling field in the spherical symmetric case, which can be expressed in terms of power series in the variable , where is the distance between the gravitational center and the orbiting particle. We then revisit the integration of the geodetic equations of the Kaluza-Klein theory with constant coupling field given by Schmidt [1990] and comment on every integration step. The resulting pseudo potential for the -motion in the ecliptic for a position is then discussed for the Einstein-Maxwell case regarding its roots, using the method of the Italian mathematician Ferrari for the determination of the four roots of a polynomial of degree four.

We then extend the investigation of the exact solution of the geodetic equations for a test particle with zero rotational moment in the Einstein-Maxwell case, which was already obtained by Schmidt [1990], by tracing the analytic continuation of this solution through the gravitational singularity of the gravitational center. For this geodetic equation we also provide a heuristic derivation that illustrates the acting of photons and gravitons between the bodies and themselves, which leads to this interaction. We find that the test particle starts on a time travel when traversing the gravitational center, yet eventually falls back to this gravitational center, thus traversing the gravitational center again. The reemerging particle thus has performed a temporal jump with respect to the entering particle. We can calculate this temporal jump exactly.

If we apply the concept of a temporal jump of a particle plunging into a gravitational center to a system where electrons repeatedly plunge in the gravitational center of a neighboring proton, we can work out the statistics of this process comprehended as a stochastic process, which leads to a normal distribution of the probability of finding a particle displaced by a specific time. For this normal distribution we can calculate the expectancy value for the total time displacement. This expectancy value is equal to the variance of the time variations of the particles induced by the stochastic process. This stochastic process is linked with an electron annihilation-recreation process, the electron is absorbed by the gravitational center of the proton and reemerges, which corresponds to an energy fluctuation with the value of two times the rest energy of the electron. The product of the time and this energy fluctuation is a constant for which we can show that it is identical with the Planck constant divided by two times pi. In this way we have derived Heisenberg's uncertainty principle for temporal and energetic fluctuations of particles, which translates to an equivalent uncertainty principle for spatial fluctuations and fluctuations of the momentum of the particles. The validity of a Heisenberg uncertainty relation with respect to position and momentum is equivalent with the existence of a Schroedinger equation governing the statistical behavior of an electron in an electron-proton system.

II. THE THEORY OF KALUZA & KLEIN

A good overview article on the topic can be found in Goenner [1984]. In summary, the basics are as follows: In a higher-dimensional space

$$R^{\sigma}_{\mu\nu\lambda} = \Gamma^{\sigma}_{\mu\lambda,\nu} - \Gamma^{\sigma}_{\mu\nu,\lambda} + \Gamma^{\sigma}_{\alpha\nu}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\sigma}_{\alpha\lambda}\Gamma^{\alpha}_{\mu\nu} \qquad (1)$$

is the Riemann tensor that prescribes how a vector changes in a curved space, when the vector is transformed with an infinitesimal coordinate transformation. Here, the functions are the so-called Christoffel symbols, the indices's range up to the dimension of the space time, double indices's indicate a summation, and the comma stands for a derivation. The Christoffel symbols are defined as

$$\Gamma^{\mu}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma\nu,\lambda} + g_{\sigma\lambda,\nu} - g_{\nu\lambda,\sigma}\right), \qquad (2)$$

where $g_{\mu\nu}$ is the metric (represented by a matrix), and $g^{\mu\nu}$ is the inverse of $g_{\mu\nu}$. Via contraction one obtains the Ricci tensor

$$R_{\mu\nu} = R^{\sigma}_{\mu\sigma\nu} \qquad (3)$$

and the curvature

$$R = g^{\mu\nu} R_{\mu\nu}. \qquad (4)$$

On the other hand, one has the Maxwell tensor

$$F_{\mu\nu} = A_{\mu,\nu} - A_{\nu,\mu} \quad (5)$$

for the electromagnetic field, which is the (four)-dimensional rotation of the vector potential A_μ of the electrodynamic field. With this tensor one can define the Maxwell scalar

$$F = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}, \quad (6)$$

where upper indices's are obtained via contraction with the inverse metric. The Lagrange density of the minimally coupled gravitational and electromagnetic fields is then given by

$$L_{minimal} = \frac{1}{2\kappa^2} R\sqrt{-g} + F\sqrt{-g}, \quad (7)$$

where g is the determinant of the metric, and $\kappa^2 = 8\pi G_N / c^4$, with Newton's gravitational constant G_N and the light velocity c. The latter factor secures that the integral over the Lagrange density and the whole space is unchanged when the integral is transformed with an arbitrary coordinate transformation. One demands that this integral becomes minimal for the real metric fields and the vector potential, which indicates that either gravitational fields produce electromagnetic fields or vice versa, i.e. the governing field equations are the Euler-Lagrange derivatives of the Lagrange density with respect to the metric fields and the components of the vector potential:

$$\frac{DL}{Df(i)} = \frac{1}{2}\left[\frac{dL}{d(f(i),\alpha,\beta)}\right]_{,\alpha,\beta} - \left[\frac{dL}{d(f(i),\alpha)}\right]_{,\alpha} + \frac{dL}{d(f(i))}, \quad (8)$$

where $f(i)$ are the $g_{\mu\nu}$ or the A_μ, and these derivatives have to be set to zero. For $L_{minimal}$ one obtains the Einstein equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_N}{c^2} T_{\mu\nu}, \quad (9)$$

where

$$T^{\mu\nu} = \frac{1}{c^2}\left(F^{\mu}_{\ \alpha} F^{\alpha\nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right) \quad (10)$$

is the energy-momentum tensor of the electromagnetic field, and the Maxwell equations for the vacuum

$$F^{\mu\nu}_{\ ;\nu} = 0, \quad (11)$$

where the semicolon denotes the covariant derivative

$$A^{\mu}_{;\nu} = \frac{\partial A^\mu}{\partial x^\nu} + \Gamma^{\mu}_{\nu\lambda} A^\lambda, \quad (12)$$

where x^ν is the coordinate ν. In this linearized form, the Kaluza-Klein theory for the unified gravitational and electromagnetic field is also called the Einstein-Maxwell theory.

In five dimensions, apart from the curvature R and a cosmological constant, there is exactly one further Lagrange density that leads to field equations of second order Lovelock [1971]. It is the so-called Gauss-Bonnet density

$$L_{Gauss-Bonnet} = \left(R_{ABCD} R^{ABCD} - 4 R_{AB} R^{AB} + R^2\right)\sqrt{-det(g_{AB})}. \quad (13)$$

For the projection of the five-dimensional metric to the space time one sets

$$(g'_{AB}) = \begin{pmatrix} g_{\mu\nu} + \psi^2 A_\mu A_\nu & \psi A_\mu \\ \psi A_\nu & \psi^2 \end{pmatrix}, \quad (14)$$

where the prime indicates independence from the coordinate x^5. This special form also secures that A_μ transforms like a U(1) gouge field. If one wants to have only a gravitational field and an electromagnetic field, one has to assume $\psi = \psi_0 = const$. (This is the original ansatz of Kaluza [1921] and Klein [1926].) Up to boundary terms, $L_{Gauss-Bonnet}$ then reduces to [Mueller-Hoissen, 1988]:

$$L_{Gauss-Bonnet} = \psi_0 \Big\{ R_{\mu\nu\kappa\lambda} R^{\mu\nu\kappa\lambda} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 +$$

$$\frac{3}{16}\psi_0^4 \left[(F_{\mu\nu} F^{\mu\nu})^2 - 2 F^{\mu}_{\ \nu} F^{\nu}_{\ \kappa} F^{\kappa}_{\ \lambda} F^{\lambda}_{\ \mu} \right] \Big\} \sqrt{-det(g_{\alpha\beta})}$$

$$-\frac{1}{2}\psi_0^3 (F_{\mu\nu} F^{\kappa\lambda} R^{\mu\nu}_{\ \ \kappa\lambda} - 4 F_{\mu\kappa} F^{\nu\kappa} R^{\mu}_{\ \nu} + F_{\mu\nu} F^{\mu\nu} R)\sqrt{-det(g_{\alpha\beta})}. \quad (15)$$

Terms in the Lagrange density that are quadratic in the curvature tensor are irrelevant for the classical dynamic [Mueller-Hoissen, 1988], and terms that are bi-quadratic in the tensor of the electromagnetic field are not considered, since those terms would lead to second order derivatives of $g_{\alpha\beta}$ entering the field equations also quadratically and the first derivatives of A_α also entering the field equations bi-quadratically. For such equations the Cauchy problem couldn't be posed properly. With the substitution $-\frac{1}{4}\gamma$ for $-\frac{1}{2}\psi_0^3$ one obtains a Lagrange density

$$L_{notminimal} = -\frac{1}{4}\gamma (F_{\mu\nu} F^{\kappa\lambda} R^{\mu\nu}_{\ \ \kappa\lambda} - 4 F_{\mu\kappa} F^{\nu\kappa} R^{\mu}_{\ \nu} + F_{\mu\nu} F^{\mu\nu} R)\sqrt{-g}, \quad (16)$$

in which the gravitational and electromagnetic fields are coupled nonlinearly, since the terms are triple products where the curvature tensor enters linearly and the Maxwell tensor enters quadratically. The full Lagrange density is then

$$L = L_{minimal} + L_{notminimal}, \quad (17)$$

which leads to the field equations

$$0 = \frac{1}{2\kappa^2}\left(R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R \right)$$

$$-\frac{1}{2}\left(F^{\alpha\rho}F_{\rho}^{\ \beta}+\frac{1}{4}g^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}\right)$$

$$-\frac{1}{4}\gamma\Big(-4F_{\kappa\mu}F_{\nu}^{\kappa}R^{\mu(\alpha\beta)\nu}-6F_{\mu\nu}F_{\lambda}^{(\alpha}R^{\beta)\lambda\mu\nu}+6F^{\alpha\mu}F^{\beta\nu}R_{\mu\nu}+$$

$$8R^{\mu(\alpha}F^{\beta)\nu}F_{\mu\nu}-2RF^{\alpha\mu}F_{\mu}^{\beta}-R^{\alpha\beta}F_{\mu\nu}F^{\mu\nu}+2\nabla^{\alpha}F_{\mu\nu}\nabla^{\beta}F^{\mu\nu}+$$

$$2\nabla_{\nu}F^{\alpha\mu}\nabla_{\mu}F^{\beta\nu}-2\nabla_{\mu}F^{\alpha\mu}\nabla_{\nu}F^{\beta\nu}+4\nabla^{(\alpha}F^{\beta)\mu}\nabla^{\nu}F_{\mu\nu}+$$

$$\left[2\nabla_{\mu}F^{\mu\kappa}\nabla^{\nu}F_{\nu\kappa}+2\nabla_{\mu}F_{\nu\kappa}\nabla^{\nu}F^{\mu\kappa}-2\nabla^{\kappa}F^{\mu\nu}\nabla_{\kappa}F_{\mu\nu}\right]g^{\alpha\beta}+$$

$$\left[\frac{3}{2}F_{\mu\nu}F^{\kappa\lambda}R^{\mu\nu}_{\ \ \kappa\lambda}-4F_{\mu\nu}F^{\kappa\nu}R^{\mu}_{\kappa}+\frac{1}{2}F_{\mu\nu}F^{\mu\nu}R\right]g^{\alpha\beta}\Big)$$

and

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\nu}}\left[\gamma F^{\kappa\lambda}\left(R^{\mu\nu}_{\ \ \kappa\lambda}-4R^{[\mu}_{\kappa}\delta^{\nu]}_{\lambda}+\left(R+\frac{1}{\gamma}\right)\delta^{[\mu}_{\kappa}\delta^{\nu]}_{\lambda}\right)\sqrt{-g}\right]=0, \quad (19)$$

where ∇ stands for the covariant derivative, δ is the Kronecker delta and the brackets are defined as follows:

$$A^{(\alpha\beta)}=\frac{1}{2}\left(A^{\alpha\beta}+A^{\beta\alpha}\right)$$

$$A^{[\alpha\beta]}=\frac{1}{2}\left(A^{\alpha\beta}-A^{\beta\alpha}\right).$$

The equations of motion for test particles follow from the variational principle

$$\delta\int\sqrt{g'_{AB}\frac{dx^{A}}{d\tau}\frac{dx^{B}}{d\tau}}d\tau != 0, \quad (22)$$

with variation with respect to $\frac{dx^{A}}{d\tau}$, and τ is the Eigen-time of the particle. When projected into the space time, x^{5} becomes a constant and thus $\frac{dx^{5}}{d\tau}=0$. The variation is then restricted to the remaining four velocities. This variational principle has the meaning that the line connecting two events has the smallest possible length. The corresponding Euler-Lagrange derivatives then yield

$$\frac{d^{2}x^{\alpha}}{d\tau^{2}}=\frac{q}{m_{0}c}F^{\alpha}_{\mu}\frac{dx^{\mu}}{d\tau}-\Gamma^{\alpha}_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau}, \quad (23)$$

where q and m_0 are the charge and the rest mass of the test particle. In (23) the first term on the right side of the equation is the correct Lorenz force term that appears in addition to the Christoffel term of the theory with pure gravitational fields.

The equations (18), (19) and (23) constitute the Kaluza-Klein theory for unified and nonlinearly coupled gravitational and electromagnetic fields.

III. THE SOLUTION OF THE FIELD EQUATIONS IN THE SPHERICALLY SYMMETRIC CASE

In the static spherically symmetric case the line element in the vacuum can be expressed as

$$ds^{2}=-e^{\nu(r)}c^{2}dt^{2}+e^{\lambda(r)}dr^{2}+r^{2}\left(d\theta^{2}+\sin^{2}\theta d\phi^{2}\right), \quad (24)$$

where t is the laboratory time of a remote observer, r, θ, and ϕ are spherically symmetric coordinates, $g_{00}=-e^{\nu(r)}$ represents the time dilation, and $g_{11}=e^{\lambda}$ represents the Lorentz contraction. For the tensor of the electromagnetic field we make the ansatz (18)

$$F_{\mu\nu}=\begin{pmatrix}0 & a(r)_{,r} & 0 & 0\\ -a(r)_{,r} & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}, \quad (25)$$

where $a(r)$ is the zero component of the vector potential or the scalar potential.

We now insert (24) and (25) in (18) and (19), which yields

$$-a'=\frac{ke^{\frac{1}{2}(\nu+\lambda)}}{r^{2}-2\gamma(e^{-\lambda}-1)}, \quad (20)$$

(21)

$$0=-2\gamma\kappa^{2}[a']^{2}(e^{\lambda}-1)-\kappa^{2}e^{\lambda}[a']^{2}r^{2}+2e^{(\lambda+\nu)}[\lambda'r+e^{\lambda}-1],$$

(27)

$$0=2\gamma\kappa^{2}[a']^{2}(e^{\lambda}-3)+\kappa^{2}e^{\lambda}[a']^{2}r^{2}+2e^{(\lambda+\nu)}[\nu'r-e^{\lambda}+1],$$

(28)

$$0=2\gamma\kappa^{2}a'[4a''-a'(3\lambda'+\nu')]+2\kappa^{2}e^{\lambda}[a']^{2}r+ e^{(\lambda+\nu)}[-2\nu''r+(\lambda'-\nu')(\nu'r+2)].$$

Equation (26) is already integrated one time and k is an integration constant (it is the Coulomb constant). If one identifies $D=e^{\frac{\nu}{2}}$, $C=e^{-\frac{\lambda}{2}}$, $Y=r$, $A=-a$, and $B_0=0$, one finds that the system of equations (26) to (29) is identical with the system of equations (2.8) to (2.11) in Mueller-Hoissen & Sippel [1988]. Since we have three unknown functions, a, ν, and λ, only three of the four equations (26) to (29) can be independent. We can choose (29) as the dependent one, and indeed Mueller-Hoissen & Sippel [1988] shows that (29) is a consequence of (26) to (28).

Equation (27) can be rearranged as

$$\kappa^{2}e^{\lambda}[a']^{2}(r^{2}-2\gamma(e^{-\lambda}-1))=2e^{(\lambda+\nu)}[\lambda'r+e^{\lambda}-1]. \quad (30)$$

If one inserts a' from (26), one gets

$$\frac{\kappa^2 k^2 e^{(\lambda+\nu)}}{[r^2 - 2\gamma(e^{-\lambda}-1)]} e^{\lambda} = 2e^{(\lambda+\nu)} e^{\lambda}[\lambda' e^{-\lambda} r + 1 - e^{-\lambda}]$$

$$\Leftrightarrow [-\lambda' e^{-\lambda} r + (e^{-\lambda}-1)][r^2 - 2\gamma(e^{-\lambda}-1)] = -\frac{\kappa^2 k^2}{2}$$

$$\Leftrightarrow (-r^2)[r(e^{-\lambda}-1)]' \left[1 - 2\gamma \frac{1}{r^3} r(e^{-\lambda}-1)\right] = \frac{\kappa^2 k^2}{2}.$$

With

$$\frac{d}{dr} = \frac{-1}{r^2} \frac{d}{d\left(\frac{1}{r}\right)} \tag{34}$$

and the definitions

$$u = \frac{1}{r},\ \ f(u) = r(e^{-\lambda}-1),\ a_1 = \frac{\kappa^2 k^2}{2} \tag{35}$$

this leads to

$$\frac{df(u)}{du} = \frac{a_1}{1 - 2\gamma u^3 f(u)}, \tag{36}$$

whereby

$$e^{\lambda} = \frac{1}{1 + uf(u)}. \tag{37}$$

For $f(u)$ we make the ansatz for a power expansion:

$$f(u) = \sum_{n=0}^{\infty} a_n u^n. \tag{38}$$

We rewrite (36) as

$$\frac{df(u)}{du} - \gamma u^3 \frac{d}{du}(f(u)^2) = \frac{\kappa^2 k^2}{2}, \tag{39}$$

and insert the expressions

$$\frac{df(u)}{du} = a_1 + 2a_2 u + 3a_3 u^2 + \sum_{n=1}^{\infty}(n+3)a_{n+3}u^{n+2} \tag{40}$$

and

$$u^3 \frac{d}{du}(f(u)^2) = \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} a_k a_{n-k}\right) n u^{n+2}, \tag{41}$$

where we used the Cauchy product for the square of a power series. This yields

$$a_1 + 2a_2 u + 3a_3 u^2 + \sum_{n=1}^{\infty}\left[(n+3)a_{n+3} - \gamma n\left(\sum_{k=0}^{n} a_k a_{n-k}\right)\right] u^{n+2} = \frac{\kappa^2 k^2}{2}. \tag{42}$$

We perform a comparison of coefficients on both sides of the equation and obtain

$$a_0 = f(0) = -2m \text{ as a free integration constant, } a_1 = \frac{\kappa^2 k^2}{2},\ a_2 = a_3 = 0$$

$$a_{n+3} = \frac{\gamma n}{n+3} \sum_{k=0}^{n} a_k a_{n-k},\ \ for\ n = 1,2,3,\ldots \tag{31}$$

This formula is as valuable as having an analytic solution, since we can get a solution as precisely as we whish by summing up enough terms in the power series. For the coefficients 4 to 13 one obtains

$$a_4 = -\frac{k^2 m \kappa^2 \gamma}{2},\ a_5 = \frac{k^4 \kappa^4 \gamma}{10} \tag{32}$$

$$a_6 = 0,\ a_7 = \frac{8k^2 m^2 \kappa^2 \gamma^2}{7} \tag{33}$$

$$a_8 = -\frac{9k^4 m \kappa^4 \gamma^2}{16},\ a_9 = \frac{k^6 \kappa^6 \gamma^2}{15}$$

$$a_{10} = -\frac{16k^2 m^3 \kappa^2 \gamma^3}{5},\ a_{11} = \frac{204k^4 m^2 \kappa^4 \gamma^3}{77}$$

$$a_{12} = -\frac{223k^6 m \kappa^6 \gamma^3}{320},\ a_{13} = \frac{k^2 \kappa^2 \gamma^3 (23k^6 \kappa^6 + 3840 m^4 \gamma)}{390}. \tag{44}$$

We note that the power series has an alternating sign, which alone already guarantees the convergence of the series. According to (37) we get

$$e^{\lambda} = \frac{1}{1 - 2mu + a_1 u^2 + O(u^5)}, \tag{45}$$

which is identical with the so-called Reissner-Nordstroem metric for $\gamma = 0$, and the parameter $2m$ can be identified with the so-called Schwarzschild radius. We note that the Reissner-Nordstroem metric has no event horizon when $m^2 < a_1$, which is fulfilled if the charge of the gravitational center is sufficiently larger than the mass of the gravitational center. In this case a test particle can approach the singularity at the origin of the gravitational center.

In order to determine the function e^{ν} we add equation (27) and (28):

$$0 = -2\gamma\kappa^2[a']^2 + e^{\lambda+\nu}[(\lambda' + \nu')r]. \tag{46}$$

If one substitutes a' from (26), it follows

$$(\lambda' + \nu')r^2 = \frac{2\gamma\kappa^2 k^2}{r^3[1 - 2\gamma \frac{1}{r^3} r(e^{-\lambda}-1)]^2}. \tag{47}$$

With the definition

$$g(u) = -[\nu(u) + \lambda(u)] \tag{48}$$

it follows

$$\frac{dg(u)}{du} = \frac{4\gamma a_1 u^3}{[1 - 2\gamma u^3 f(u)]^2}. \tag{49}$$

The boundary condition is $g(0) = 0$, since asymptotically for large r it should hold $\nu = -\lambda$ as in the Reissner-Nordstroem case. Because of $\frac{df(u)}{du} = \frac{a_1}{1 - 2\gamma u^3 f(u)}$ we have

$$\frac{dg(u)}{du} = \frac{4\gamma u^3}{a_1}\left[\frac{df(u)}{du}\right]^2. \tag{50}$$

Because of

$$\frac{df(u)}{du} = \sum_{n=1}^{\infty} a_n n u^{n-1} = \sum_{n=0}^{\infty} a_{n+1}(n+1)u^n \tag{51}$$

we get the Cauchy product

$$\left[\frac{df(u)}{du}\right]^2 = \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} a_{k+1}(k+1)a_{n-k+1}(n-k+1)\right]u^n$$

$$\Rightarrow \frac{dg(u)}{du} = \frac{4\gamma}{a_1}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1}\right]u^{n+3}$$

$$\Rightarrow g(u) = \underbrace{g(0)}_{=0} + \frac{4\gamma}{a_1}\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}(k+1)(n-k+1)a_{k+1}a_{n-k+1}\right]\frac{u^{n+4}}{(n+4)}, \tag{54}$$

where the coefficients a_n are taken from (43). If one knows $g(u)$, one obtains

$$e^{\nu} = e^{-g}\,\frac{1}{e^{\lambda}}. \tag{55}$$

Finally, one obtains from (26)

$$\frac{da(u)}{du} = \frac{k e^{-\frac{1}{2}g(u)}}{1-2\gamma u^3 f(u)} = \frac{k}{a_1}e^{-\frac{1}{2}g(u)}\frac{df(u)}{du}. \tag{56}$$

In equation (56) it is

$$e^{-\frac{1}{2}g(u)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}(g(u))^n$$

$$g(u) = \sum_{m=0}^{\infty} b_m u^m \quad with \quad b_0 = b_1 = b_2 = b_3 = 0$$

$$and \quad b_m = \frac{4\gamma}{a_1 m}\sum_{k=0}^{m-4}(k+1)(m-k-3)a_{k+1}a_{m-k-3} \quad for \quad m = 4,5,6,\ldots$$

$$according\,to \quad (54).$$

Thus, we have

$$e^{-\frac{1}{2}g(u)} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\sum_{k_1=0}^{\infty}\left({}^{(1)}\sum_{k_2=0}^{k_1}\left({}^{(2)}\cdots\sum_{k_{n-3}}^{k_{n-4}}\left({}^{(n-3)}\sum_{k_{n-2}=0}^{k_{n-3}}\left({}^{(n-2)}\sum_{k_{n-1}=0}^{k_{n-2}}\left({}^{(n-1)}\left({}^{(n)}\sum_{k_n=0}^{k_{n-1}} b_{k_n}b_{k_{n-1}-k_n}\right)^{(n)} b_{k_{n-2}-k_{n-1}}\right)^{(n-1)} b_{k_{n-3}-k_{n-2}}\right)^{(n-2)} b_{k_{n-4}-k_{n-3}}\right)^{(n-3)}\cdots\right)^{(2)} b_{k_1-k_2}\right)^{(1)} \times u^{k_1},$$

where we used n-times Cauchy products. Again with $\frac{df(u)}{du} = \sum_{k_0=0}^{\infty}(k_0+1)a_{k_0+1}u^{k_0}$ we get

$$e^{-\frac{1}{2}g(u)}\frac{df(u)}{du} = \sum_{m=0}^{\infty}\left({}^{(0)}\sum_{k_1=0}^{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\left({}^{(1)}\sum_{k_2=0}^{k_1}\left({}^{(2)}\cdots\sum_{k_{n-3}}^{k_{n-4}}\left({}^{(n-3)}\sum_{k_{n-2}=0}^{k_{n-3}}\left({}^{(n-2)}\sum_{k_{n-1}=0}^{k_{n-2}}\left({}^{(n-1)}\left({}^{(n)}\sum_{k_n=0}^{k_{n-1}} b_{k_n}b_{k_{n-1}-k_n}\right)^{(n)} b_{k_{n-2}-k_{n-1}}\right)^{(n-1)} b_{k_{n-3}-k_{n-2}}\right)^{(n-2)} b_{k_{n-4}-k_{n-3}}\right)^{(n-3)}\cdots\right)^{(2)} b_{k_1-k_2}\right)^{(1)} \tag{52}$$

$$\times (m-k_1+1)a_{m-k_1+1}\Big)^{(0)} u^m, \tag{53}$$

where we applied another Cauchy product. Via integration we obtain

$$a(u) = \frac{k}{a_1}\sum_{m=0}^{\infty}\left({}^{(0)}\sum_{k_1=0}^{m}\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}\left({}^{(1)}\sum_{k_2=0}^{k_1}\left({}^{(2)}\cdots\sum_{k_{n-3}}^{k_{n-4}}\left({}^{(n-3)}\sum_{k_{n-2}=0}^{k_{n-3}}\left({}^{(n-2)}\sum_{k_{n-1}=0}^{k_{n-2}}\left({}^{(n-1)}\left({}^{(n)}\sum_{k_n=0}^{k_{n-1}} b_{k_n}b_{k_{n-1}-k_n}\right)^{(n)} b_{k_{n-2}-k_{n-1}}\right)^{(n-1)} b_{k_{n-3}-k_{n-2}}\right)^{(n-2)} b_{k_{n-4}-k_{n-3}}\right)^{(n-3)}\cdots\right)^{(2)} b_{k_1-k_2}\right)^{(1)}$$

$$\times (m-k_1+1)a_{m-k_1+1}\Big)^{(0)}\frac{u^{m+1}}{(m+1)}, \tag{57}$$

where we have set $a(0) = 0$, since the Coulomb potential shall vanish for $r \to \infty$.

IV. THE EQUATIONS OF MOTION FOR TEST PARTICLES

If one inserts (24) and (25) in (23), one obtains

$$c\ddot{t} = (-\nu')c\dot{t}\dot{r} + (-qa'e^{-\nu})\dot{r}, \tag{58}$$

$$\ddot{r} = \left(-\frac{1}{2}e^{\nu-\lambda}\nu'\right)c^2[\dot{t}]^2 + \left(-\frac{1}{2}\lambda'\right)[\dot{r}]^2 + (re^{-\lambda})[\dot{\theta}]^2 +$$
$$(\sin^2\theta re^{-\lambda})[\dot{\phi}]^2 + (-qa'e^{-\lambda})c\dot{t},$$
$$\ddot{\theta} = \left(-\frac{2}{r}\right)\dot{r}\dot{\theta} + (\sin\theta\cos\theta)[\dot{\phi}]^2,$$
$$\ddot{\phi} = \left(-\frac{2}{r}\right)\dot{r}\dot{\phi} + (-2\cot\theta)\dot{\theta}\dot{\phi} \tag{59}$$

for the geodetic equations. (The geodetic equations have been derived even for the more general rotational symmetric case [see, e.g., Carter, 1968; Misner, 1973, equations 33.32a-d with the solutions 33.37a-d]. However, the two systems of equations can be linked with an arbitrary coordinate transformation. Thus, it is non-trivial to proof the equivalence of both systems. We have checked our result with computer algebra.)

We multiply (62) with e^{ν}:

$$e^{\nu} c\ddot{t} = -e^{\nu}\nu' \dot{r} c\dot{t} - q a' \dot{r} \qquad (66)$$

It is

$$\nu' \dot{r} = \frac{d\nu}{dr}\frac{dr}{d\tau} = \frac{d\nu}{d\tau}$$

$$a' \dot{r} = \frac{da}{dr}\frac{dr}{d\tau} = \frac{da}{d\tau}$$

and thus

$$e^{\nu}\nu' \dot{r} = e^{\nu}\frac{d\nu}{d\tau} = \frac{de^{\nu}}{d\tau}$$

$$\Rightarrow\ e^{\nu} c\ddot{t} + e^{\nu}\nu' \dot{r} c\dot{t} = e^{\nu}\frac{dc\dot{t}}{d\tau} + \frac{de^{\nu}}{d\tau}c\dot{t} = \frac{de^{\nu}c\dot{t}}{d\tau} = -q\frac{da}{d\tau}$$

$$\Leftrightarrow\ \frac{d(e^{\nu}c\dot{t} + qa)}{d\tau} = 0$$

$$\Rightarrow\ e^{\nu}c\dot{t} + qa = \beta,$$

where the integration constant β can be interpreted as energy parameter (to be more precise, β^2 is the energy per mass unit of the test particle).

Equation (63) can be replaced with the definition of the line element (24), since one geodetic equation or a combination must be identical with that definition. (This can also be shown with the help of the field equations (26) - (28) directly.) If one considers that for the chosen signature of the metric $ds^2 = -c^2 d\tau^2$ holds, one obtains after division by $d\tau^2$:

$$-c^2 = -e^{\nu}c^2[\dot{t}]^2 + e^{\lambda}[\dot{r}]^2 + r^2\left([\dot{\theta}]^2 + \sin^2\theta[\dot{\phi}]^2\right) \qquad (73)$$

The coordinate system can always be chosen such that for a starting value τ_0 $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$ holds. However, $\theta = \frac{\pi}{2}$ is a solution of (64) for all τ, and because the start values determine the solution unambiguously, this is exactly the solution that is searched for. Thus, the trajectories are planar:

$$\theta = \frac{\pi}{2}. \qquad (74)$$

With this, also (65) simplifies, and we obtain

$$\ddot{\phi} = \left(-\frac{2}{r}\right)\dot{r}\dot{\phi}$$

$$\Leftrightarrow\ \frac{\ddot{\phi}}{\dot{\phi}} = -2\frac{\dot{r}}{r}$$

$$\Leftrightarrow\ \frac{d}{d\tau}\left(\log(\dot{\phi})\right) = -2\frac{d}{d\tau}\left(\log r\right) = -\frac{d}{d\tau}\left(\log(r^2)\right)$$

$$\Leftrightarrow\ 0 = \frac{d}{d\tau}\left(\log(\dot{\phi}) + \log(r^2)\right) = \frac{d}{d\tau}\left(\log(\dot{\phi}r^2)\right)$$

$$\Leftrightarrow\ const = \log(\dot{\phi}r^2)$$

$$\Leftrightarrow\ \dot{\phi} = \frac{\alpha}{r^2}, \qquad (67)$$

where the integration constant α can be interpreted as a rotational momentum parameter. More precisely, (68) α is the rotational momentum per mass unit of the test particle.

With the solutions (72), (74) and (80), (73) yields:

$$-c^2 = -e^{-\nu}[\beta - qa]^2 + e^{\lambda}[\dot{r}]^2 + \frac{\alpha^2}{r^2}. \qquad (81)$$

(69)

We again introduce the variable $u = \frac{1}{r}$, for which it follows (70)

$$\dot{r} = \frac{dr}{d\tau} = \alpha u^2\frac{d}{d\phi}\left(\frac{1}{u}\right) = -\alpha\frac{du}{d\phi}. \qquad (82)$$

(71) (72)

It follows

$$\left[\frac{du}{d\phi}\right]^2 + \left[\left(u^2 + \frac{c^2}{\alpha^2}\right)e^{-\lambda(u)} - \frac{1}{\alpha^2}[\beta - qa(u)]^2 e^{-(\nu(u)+\lambda(u))}\right] = 0 \qquad (83)$$

for $\alpha \neq 0$. In (83),

$$V_{pseudo}(u,\beta,\alpha) = \left(u^2 + \frac{c^2}{\alpha^2}\right)e^{-\lambda(u)} - \frac{1}{\alpha^2}[\beta - qa(u)]^2 e^{-(\nu(u)+\lambda(u))} \qquad (84)$$

can be seen as a energy- and rotational momentum dependent pseudo potential, in which a particle at a position u moves with the trajectory parameter ϕ.

In the case $\alpha = 0$ we obtain from (81):

$$\dot{r} = \pm\sqrt{e^{-(\nu+\lambda)}\left([\beta - qa]^2 - c^2 e^{\nu}\right)}, \qquad (85)$$

where the negative sign belongs to a falling particle.

For the further discussion we look at the case with $\gamma = 0$, i. e. the Einstein-Maxwell case, and where the electromagnetic interaction is dominant. Since the Gauss-Bonnet Lagrange density for the non-linear coupling consists of triple tensor products between a quadratic term in the Maxwell tensor and a curvature tensor, whereas the Maxwell scalar of the Einstein-Maxwell theory is quadratic in the Maxwell tensor alone, the non-linear coupling terms are second order correction terms to (75) the terms stemming from the Maxwell scalar. So in this case the solutions of the Einstein-Maxwell theory are very close to the full solutions of the non-linear theory. One finds that (76) within Einstein-Maxwell's theory one can make substantial analytic advances, thus capturing the essence of the physics of the theory.

For $\gamma = 0$ the solution of the field equations is the so-called Reissner-Nordstroem metric $e^{\nu} = e^{-\lambda} = 1 - 2mu + \frac{\kappa^2 k^2}{2} u^2$ and $a = ku$. The pseudo-potential (84) then reduces to

$$V_{pseudo}(u,\alpha,\beta) = \left(u^2 + \frac{c^2}{\alpha^2}\right)e^{\nu} - \frac{1}{\alpha^2}[\beta - qa(u)]^2 \quad (86)$$

and the radial geodetic equation (85) to

$$\dot{r} = \pm\sqrt{[\beta - qa]^2 - c^2 e^{\nu}}. \quad (87)$$

One sees that the pseudo potential is a polynomial of fourth order in u now. The specifications are

$$V_{pseudo}(u,\alpha,\beta) = a_1(u^4 + \bar{a}u^3 + \bar{b}u^2 + \bar{c}u + \bar{d}), \quad (88)$$

where

$$a_1 = \frac{\kappa^2 k^2}{2},$$
$$\bar{a} = -\frac{2m}{a_1},\ \bar{b} = \frac{\alpha^2 + a_1 c^2 - q^2 k^2}{a_1 \alpha^2},$$
$$\bar{c} = \frac{2[\beta q k - mc^2]}{a_1 \alpha^2},\ \bar{d} = \frac{c^2 - \beta^2}{a_1 \alpha^2}.$$

The four zeros of the potential (88) can be calculated exactly following the method of the Italian mathematician Ferrari [see, e.g., Reinhard & Soeder, 1984]. This method goes as follows. In the equation

$$u^4 + \bar{a}u^3 + \bar{b}u^2 + \bar{c}u + \bar{d} = 0 \quad (90)$$

on substitutes $u = z - \frac{\bar{a}}{4}$. This yields

$$z^4 + \tilde{p}z^2 + \tilde{q}z + \tilde{r} = 0,$$

$$where\ \ \tilde{p} = \bar{b} - \frac{3}{8}\bar{a}^2,\ \tilde{q} = \bar{c} - \frac{\bar{a}\bar{b}}{2} + \frac{1}{8}\bar{a}^3,$$
$$\tilde{r} = \bar{d} - \frac{\bar{a}\bar{c}}{4} + \frac{1}{16}\bar{a}^2\bar{b} - \frac{3}{256}\bar{a}^4.$$

The equation (100) can be rewritten as

$$(z^2 + P)^2 - (Qz + R)^2 = 0,$$

$$where\ \ 2P - Q^2 = \tilde{p},\ -2QR = \tilde{q},\ P^2 - R^2 = \tilde{r}.$$

This means that (93) is fulfilled if

$$z^2 + P = Qz + R\ \ or\ \ z^2 + P = -Qz - R \quad (95)$$

is fulfilled. In the case $\tilde{q} = 0$, the biquadratic equation $z^4 + \tilde{p}z^2 + \tilde{r} = 0$ can be reduced to quadratic equations $z^2 = -\frac{\tilde{p}}{2} \pm \sqrt{\left(\frac{\tilde{p}}{2}\right)^2 - \tilde{r}}$

$$\Rightarrow\ u_{0/1/2/3} = -\frac{\bar{a}}{4} \pm \sqrt{-\frac{\tilde{p}}{2} \pm \sqrt{\left(\frac{\tilde{p}}{2}\right)^2 - \tilde{r}}}.$$

In the case $\tilde{q} \neq 0$ one has to determine a solution (P,Q,R) of the system of equations $2P - Q^2 = \tilde{p}$ and $-2QR = \tilde{q}$ and $P^2 - R^2 = \tilde{r}$ first. This system can be rewritten as:

$$(1f)\ \ Q^2 = \frac{\tilde{q}^2}{4(P^2 - \tilde{r})},\ \ (2f)\ \ R^2 = P^2 - \tilde{r},\ \ (3f)\ \ QR = -\frac{\tilde{q}}{2},$$

$$(4f)\ \ P^3 - \frac{\tilde{p}}{2}P^2 - \tilde{r}P + \frac{\tilde{p}\tilde{r}}{2} - \frac{1}{8}\tilde{q}^2 = 0,$$

whereby $(3f)$ is basically a sign check. For a solution P of $(4f)$ one chooses from the solutions of $(1f)$ and $(2f)$ a pair (Q,R) that satisfies $(3f)$, and inserts (P,Q,R) in $z^2 + P = Qz + R$ or $z^2 + P = -Qz - R$. The latter are two quadratic equations that have the solutions

$$u_{0/1} = -\frac{\bar{a}}{4} + \frac{Q}{2} \pm \sqrt{\left(\frac{Q}{2}\right)^2 + R - P}, \quad (89)$$
$$u_{2/3} = -\frac{\bar{a}}{4} - \frac{Q}{2} \pm \sqrt{\left(\frac{Q}{2}\right)^2 - R - P},$$

where we resubstituted $u = z - \frac{\bar{a}}{4}$.

For the solution of $(4f)$ we do the following: We set

$$\hat{a} = -\frac{\tilde{p}}{2},\ \hat{b} = -\tilde{r},\ \hat{c} = \frac{\tilde{p}\tilde{r}}{2} - \frac{1}{8}\tilde{q}^2 \quad (91)$$
$$(4f) \Leftrightarrow\ P^3 + \hat{a}P^2 + \hat{b}P + \hat{c} = 0$$

We substitute $P = \hat{z} - \frac{\hat{a}}{3}$ and obtain

$$\hat{z}^3 + \hat{p}\hat{z} + \hat{q} = 0 \quad (92)$$

$$with\ \ \hat{p} = \hat{b} - \frac{1}{3}\hat{a}^2,\ \hat{q} = \frac{2}{27}\hat{a}^3 - \frac{1}{3}\hat{a}\hat{b} + \hat{c}. \quad (93)$$

For $\hat{p} = 0$ one gets $\hat{z} = \sqrt[3]{-\hat{q}}$ or $P = -\frac{\hat{a}}{3} + \sqrt[3]{-\hat{q}}$, (94)

which is in fact three solutions, since $\sqrt[3]{-1}$ has three solutions, the real of which is -1.

In the case $\hat{p} \neq 0$, the substitution $\hat{z} = \hat{u} + \hat{v}$ yields

$$\hat{u}^3 + \hat{v}^3 + \hat{q} + (3\hat{u}\hat{v} + \hat{p})(\hat{u} + \hat{v}) = 0. \quad (102)$$

If $(\hat{u},\hat{v})$ is a solution of

$$\hat{u}^3 + \hat{v}^3 + \hat{q} = 0\ \ and\ \ 3\hat{u}\hat{v} + \hat{p} = 0,$$

(103)

then $\hat{z} = \hat{u} + \hat{v}$ is a solution of (100). (103) can be rewritten as

$$\hat{u}^3 = -\frac{\hat{q}}{2} + \sqrt{\frac{1}{4}\hat{q}^2 + \frac{1}{27}\hat{p}^3} \quad and \quad 3\hat{u}\hat{v} + \hat{p} = 0$$

$$with \quad \hat{p} = \hat{b} - \frac{1}{3}\hat{a}^2, \ \hat{q} = \frac{2}{27}\hat{a}^3 - \frac{1}{3}\hat{a}\hat{b} + \hat{c}. \qquad (104)$$

With this the task has been reduced to the solution of a pure cubic equation of the form

$$\hat{u}^3 = \hat{k}, \quad where \quad \hat{k} \in \mathbb{C}. \qquad (105)$$

(106)

This equation has three solutions in the body of the complex numbers $\mathbb{C}$:

$$\hat{u}_1 = \sqrt[3]{\hat{k}}, \ \hat{u}_2 = \varepsilon \hat{u}_1, \ \hat{u}_3 = \varepsilon^2 \hat{u}_1, \quad where \quad \varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i. \qquad (107)$$

The equation $3\hat{u}\hat{v} + \hat{p} = 0$ then yields values $\hat{v}_1$, $\hat{v}_2$, and $\hat{v}_3$ for $\hat{u}_1$, $\hat{u}_2$, and $\hat{u}_3$. One obtains solutions of (100) as:

$$\hat{z}_1 = \hat{u}_1 + \hat{v}_1 \quad with \quad \hat{u}_1 = \sqrt[3]{-\frac{\hat{q}}{2} + \sqrt{\frac{1}{4}\hat{q}^2 + \frac{1}{27}\hat{p}^3}} \quad and \quad \hat{v}_1 = -\frac{\hat{p}}{3\hat{u}_1},$$

$$\hat{z}_2 = \hat{u}_2 + \hat{v}_2 = \varepsilon\hat{u}_1 + \varepsilon^2\hat{v}_1, \ \hat{z}_3 = \hat{u}_3 + \hat{v}_3 = \varepsilon^2\hat{u}_1 + \varepsilon\hat{v}_1 \quad (\varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2} i),$$

$$and\ thus \quad P_1 = \hat{z}_1 - \frac{\hat{a}}{3}, \ P_2 = \hat{z}_2 - \frac{\hat{a}}{3}, \ and \ P_3 = \hat{z}_3 - \frac{\hat{a}}{3}. \qquad (108)$$

V. The solution of the radial geodetic equation in Einstein-Maxwell's theory

We have derived the radial geodetic equation in Einstein-Maxwell's theory (87) from theory rigorously. We now want to parallel this derivation with a chain of heuristic arguments. This argumentation in terms of simple physical pictures and processes gives us a deeper insight in how natural physical processes work down to the level of photon and graviton interactions. Strictly speeking, one would have to build up the interaction in terms of summations over Feynman graphs, and the classical fields would be the envelopes of the Hamiltonian of these summations, but here we have a seldom case where the fields can be treated in a purely classical and smooth sense and integrated as such.

We start our heuristic derivation with noting that the gravitational center is placed statically in the origin of the coordinate system, and it influences the motion of the orbiting object with gravitational and electromagnetic forces. It is found that in the vicinity of the object, where these forces do not vary much, the total force F on the object is proportional to the acceleration $\ddot{r}$ of the object. The proportionality factor m_0 is called the mass of the object.

$$F = m_0 \ddot{r} \qquad (109)$$

This is, in principle, not very difficult to comprehend, since it simply states that if you want to increase the velocity of an object two or multiple times, you need to increase the force on the object two or multiple times. In a spherically symmetric system, F is a function of r alone. From multiplying equation (109) with $\dot{r}$, we get

$$F(r) \cdot \dot{r} = m_0 \cdot \ddot{r} \cdot \dot{r}$$

$$\Leftrightarrow -\frac{d}{d\tau}\left(E(r)\right) = \frac{d}{d\tau}\left(\frac{m_0}{2}\dot{r}^2\right),$$

where $E(r)$ is a function with $-dE(r)/dr = F(r)$. Please remember the chain rule and the product rule. $E(r)$ is called the potential energy function of the force field F. The other term, $m_0/2\dot{r}^2$, is called the kinetic energy of the object. From (110) it follows

$$\frac{m_0}{2}\dot{r}^2 + E(r) = const = \frac{m_0}{2} v_{tot}^2, \qquad (111)$$

where v_{tot} is a constant velocity, which can be understood as a conserved energy parameter for the full system. One of the most important interaction forces is the electromagnetic interaction force. This interaction can be understood as an exchange of photons between the charged gravitational center and the object, where these photons travel along bipolar field lines from the gravitational center toward the object. Yet, those photons traveling along bent field lines in this bipolar field can be thought to be projected on the straight line that links the charged gravitational center and the object. The object catches these photons and remits them immediately. Then, the photons travel back to the charged gravitational center, where they are caught and remitted likewise. After this second reemission, the process repeats. The photons themselves, or better to say the whole photon field, since there are many, are disturbances in the space-time continuum, which consist of periodic small time-dilation and Lorentz-contraction effects, traveling like a wave through the continuum. The bouncing back and forth of these photons, thus, creates an association between the charged gravitational center and the object, which is identical with the force. Since the electromagnetic wave has a chirality, which means that the wave vector pointing from the direction of the photon propagation to the location of the disturbance in the space-time continuum rotates either clockwise or anticlockwise around that direction of propagation, the electromagnetic interaction force can either be attractive or repulsive. The force is attractive, when the photons being sent out from the charged gravitational center have the opposite chirality than the photons sent out from the object, since in this case the oppositely traveling photons engage each other maximally, thus pulling the spirals of chirality tightly together. In the case that the photons being sent out have the same chirality, they virtually decouple, and the spirals of chirality act comparable with compressed spiral springs, which drive the charged gravitational center and the object apart from each other. Depending on the kind of chirality the charged gravitational center and the object are sending out, they are denoted as positive or negative charged.

A photon field has another important property. When the photons in the field come very close to each other, the time-dilation and Lorentz-contraction effects of which they consist of have the tendency to cluster together. This creates a force in the photon field itself that distracts a photon from flying straight on, and, instead, forces it into an orbit around the clustered other photons. Thus, an entity emerges, that consists of photons orbiting each other. The radius of this rotating object is ever shrinking, until it has reached the size of a very small point. This entity is then called an elementary particle. It is not easy to shift the location of such an elementary particle, since such a shift requires the shift of all the space-time fluctuations involved to another place. Hence, the elementary particle has a natural resistance to its replacement. This resistance or inertia force is called the gravitational force of the object. When two objects come close to each other, there is still the tendency that both objects would like to cluster together like sticky meat balls in a bowl of a kitchen. Therefore, the gravitational force between the two objects is always attractive, regardless of the charge they have. The gravitational force is still there, when one or both of the objects are uncharged, since being uncharged simply means that two oppositely charged particles orbit each other very closely, whereby the absolute value of the both charges is identical. This gravitational force, like the electromagnetic force, can also be described with a field consisting of exchanged quantum, which are called gravitons in this case. These gravitons, however, have the quality that they never can repulse each other.

We now return to the discussion of the object with mass m_0, on which the observer travels. In all, this object can be understood as an entity of many particles, in all of which photons are orbiting each other. Since the velocity of these photons is always the light velocity c, they contribute with a value $m_0c^2/2$ to the balance of the dynamical energies:

$$\frac{m_0}{2}\dot{r}^2 + E(r) = \frac{m_0}{2}v_{tot}^2 - \frac{m_0}{2}c^2 \tag{112}$$

Here, the term $m_0c^2/2$ is subtracted from $m_0v_{tot}^2/2$, since this energy amount is not available to be fed into an increase of the kinetic energy $m_0\dot{r}^2/2$ of the object.

If e and Q are the charges of the object and the charged gravitational center, which can be understood as the source strength for the number of photons that both entities can emit, then

$$F_{elec} = \frac{Qe}{4\pi\varepsilon_0 r^2} \tag{113}$$

is the electromagnetic force between the charged gravitational center and the object, where the factor $4\pi\varepsilon_0$ with the so-called permissivity ε_0 is a scaling factor that takes the procedure of charge measurements into account. Since the electromagnetic force is inflicted with radially propagating photons that intersect with a sphere of area $A = 4\pi r^2$ at a distance r, and the number of such intersecting photons is conserved, in twice the distance only a quarter of these intersecting photons cover the same area on the surface of the sphere. Thus, the force must decrease proportional to $1/r^2$ exactly. There are no measurements thus far that have revealed any deviations from that law. We note that F_{elec} is negative, i.e. directed toward the charged gravitational center, if Q and e have opposite sign. For the energy E_{elec} of the electromagnetic field that has the property $-dE_{elec}/dr = F_{elec}$, we get

$$E_{elec} = \frac{Qe}{4\pi\varepsilon_0 r}, \tag{114}$$

since $-d(1/r)/dr = 1/r^2$. Now, $p = m_{dyn}c$ is the relativistic momentum of an object. Here, m_{dyn} is the dynamical mass of an object, which depends on the object's velocity and on its position in the system. Yet, we do not have to bother much about this dynamical mass, since we measure everything in the rest frame of the object. Hence, $p = m_0c$. Because of the conservation of the momentum, the electromagnetic interaction field travels with the same momentum as the object, seen from the point of view of the object. Thus, $p = m_0c$ is also the momentum of the electromagnetic field. $v = E/p$ is the relativistic dynamical velocity of a field. Thus, we have

$$v_{elec} = \frac{E_{elec}}{p} = \frac{1}{m_0c}\frac{Qe}{4\pi\varepsilon_0 r} \tag{115}$$

as the dynamical velocity of the electromagnetic field, with which the object and the charged gravitational center interacts. With this we get

$$\frac{m_0}{2}\dot{r}^2 + E(r) = \frac{m_0}{2}\left(v_{tot} + \frac{1}{m_0c}\frac{Qe}{4\pi\varepsilon_0 r}\right)^2 - \frac{m_0}{2}c^2 \tag{116}$$

as the energy balance of the system, which includes the electromagnetic interaction between the object and the charged gravitational center. Here, the dynamical velocity v_{elec} has to be added to v_{tot}, since this velocity adds to the ability of the system to feed energy into the kinetic energy $m_0\dot{r}^2/2$ of the object. Note that for an attractive force, i.e. when Qe is negative, v_{tot} is actually diminished, such that $\dot{r}$ can only get smaller values at the same distance r. This

concept of adding the dynamical velocity of the electromagnetic field to the total dynamical velocity of the system is known as the principle of the "covariant derivative" in particle physics, the science that deals with the motion of elementary particles. This principle expresses the fact that the overall effect of an electromagnetic force field is such that the increase of the total velocity of the system is retarded with the interaction field. The situation is similar to a horse carrying a wagon, where the versatility of the horse is diminished with the wagon which it has to drag along.
The gravitational force between the object and the gravitational center, then, is

$$F_{grav} = -\frac{G_N m_0 M_\varepsilon}{r^2}, \qquad (117)$$

where G_N is Newton's gravitational constant and M_ε is the mass of the gravitational center. This force law is quite similar to the force law for the electromagnetic interaction, since here, too, the interaction is inflicted with the interchange of gravitational field quantums, the gravitons. Similarly, we get

$$E_{grav} = -\frac{G_N m_0 M_\varepsilon}{r} \qquad (118)$$

for the gravitational energy, which satisfies $-dE_{grav}/dr = F_{grav}$. With the inclusion of the gravitational interaction between the object and the gravitational center, the energy balance reads

$$\frac{m_0}{2}\dot{r}^2 + E(r) = \frac{m_0}{2}\left(v_{tot} + \frac{1}{m_0 c}\frac{Qe}{4\pi\varepsilon_0 r}\right)^2 - \frac{m_0}{2}c^2 + \frac{G_N m_0 M_\varepsilon}{r}$$

$$= \frac{m_0}{2}\left(\left[v_{tot} + \frac{1}{m_0 c}\frac{Qe}{4\pi\varepsilon_0 r}\right]^2 - c^2\left[1 - \frac{2G_N M_\varepsilon}{c^2 r}\right]\right)$$

Here, the energy function E_{grav} is actually retrieved from the general energy function $E(r)$ on the left side of the balance, i.e. it enters the balance linearly. This is contrary to the entry of the energy function E_{elec} in the balance as a squared term. The difference, basically, is related to the fact that in the electromagnetic interaction the photons have to interact with two bodies in order to establish the connection. Contrary to this, in the case of the gravitational interaction, the interaction is already established when the graviton, coming from one object, reaches the other body and sticks to it. Thus, the gravitational interaction is quite a sticky thing, and its effect merely is retrieving energy from the energy reservoir that can feed the kinetic energy of the object.
It is still possible that the gravitational field of the object interacts with the electromagnetic field of the gravitational center. For this interaction, the dynamical self energy of the electromagnetic field of the charged gravitational center is

$$E_{Efield} = -\frac{Q^2}{4\pi\varepsilon_0 r}\sqrt{4\pi} \qquad (120)$$

and the dynamical self energy of the gravitational field of the object is

$$E_{Gfield} = -\frac{G_N m_0^2}{r}\sqrt{4\pi} \qquad (121)$$

The factor $\sqrt{4\pi}$ stems from the fact, that the self-interacting fields are monopole fields, where radial field lines are stretched out on the surface of a sphere $A = 4\pi r^2$. This is different to a field that interacts between two bodies, which is bipolar, and where the field lines can be thought of being distributed only along the straight connection line between the bodies. Hence, one has to substitute $r' = \sqrt{A} = \sqrt{4\pi}r \Leftrightarrow r = r'/\sqrt{4\pi}$ in the expressions of the dynamical field energies. This gives equations (120) and (121), if one drops the prime again.
Since the electromagnetic field of the gravitational center moves with the momentum $p = m_0 c$ with respect to the object, and $p = m_0 c$ is also the momentum of the gravitational field of the object in the rest frame of the object, we get

$$v_{Efield} = -\frac{1}{m_0 c}\frac{Q^2}{4\pi\varepsilon_0 r}\sqrt{4\pi}; v_{Gfield} = -\frac{1}{m_0 c}\frac{G_N m_0^2}{r}\sqrt{4\pi} \qquad (122)$$

for the dynamical velocities of the electromagnetic field of the charged gravitational center and the gravitational field of the object. Thus, we get

$$E_{dynEG} = \frac{m_0}{2}v_{Efield}v_{Gfield} = \frac{m_0}{2}\frac{1}{m_0 c}\frac{Q^2}{4\pi\varepsilon_0 r}\sqrt{4\pi}\cdot\frac{1}{m_0 c}\frac{G_N m_0^2}{r}\sqrt{4\pi}$$

$$= \frac{m_0}{2}c^2\frac{8\pi G_N}{c^4}\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{r^2} \qquad (119)$$

as the dynamic energy of the interacting electromagnetic field of the charged gravitational center and the gravitational field of the object. Note that this expression has a similar structure as $E_{dyn} = m_0\dot{r}^2/2$ for the dynamical or kinetic energy of the object, where the velocity field of the object interacts with itself so to say.
The energy balance now reads

$$\frac{m_0}{2}\dot{r}^2 + \frac{m_0}{2}c^2\frac{8\pi G_N}{c^4}\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{r^2} = \frac{m_0}{2}\left(\left[v_{tot} + \frac{1}{m_0 c}\frac{Qe}{4\pi\varepsilon_0 r}\right]^2 - c^2\left[1 - \frac{2G_N M_\varepsilon}{c^2 r}\right]\right)$$

$$\Leftrightarrow \frac{m_0}{2}\dot{r}^2 = \frac{m_0}{2}\left(\left[v_{tot} - \frac{1}{m_0 c}\frac{(-Q)e}{4\pi\varepsilon_0 r}\right]^2 - c^2\left[1 - \frac{2G_N M_\varepsilon}{c^2 r} + \frac{8\pi G_N}{c^4}\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{r^2}\right]\right) \qquad (124)$$

Note that this time E_{dynEG} enters the balance linearly, too, since it is a single sided interaction again, with sticky gravitons hitting photons. The term

$1-\frac{2G_N M_\varepsilon}{c^2 r}+\frac{8\pi G_N}{c^4}\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\frac{1}{r^2}$ is known as the gravitational potential or metric field of the charged gravitational field of a charged gravitational center. It has been rigorously derived from the general relativistic field equations of Albert Einstein by Reissner and Nordstroem. In the case that there is no charge of the gravitational center, this gravitational potential vanishes at a radius $r_{Schwarz}=2G_N M_\varepsilon /c^2$. This radius is called the Schwarzschild radius. In the charge-free case the gravitational center shields itself with a sphere of this radius, and the time flow of the outside world comes to a standstill at this location. Also note that there is no term for the interaction between the electromagnetic field of the object and the gravitational field of the gravitational center, since in the rest frame of the object the electromagnetic field of the object vanishes. For the same reason there is no term for the self energy of the electromagnetic field of the object. There are also no terms for the self energy of the electromagnetic field and the gravitational field of the gravitational center, since from the point of view of the object these fields are static and do not contribute to the dynamic energies.

Equation (124) can also be derived, of course, by rigorously solving Albert Einstein's general relativistic field equations and equations of motion for a spherically symmetric static charged system with mass that is orbited with an object which has mass and charge (compare with (87)). Yet, this is a very cumbersome procedure. Here, I have given you a physical interpretation of each term that occurs in this equation of motion. Note that in the case of a very close approach of the object and the charged gravitational center the electromagnetic field - gravitational field interaction and the electromagnetic field interaction become dominant, since these interaction terms depend on $1/r^2$. By rearranging terms, we get

$$\dot{r}^2=-\underbrace{\left[c^2-v_{tot}^2\right]}_{C}+\underbrace{2\left[G_N M_\varepsilon-v_{tot}\frac{(-Q)e}{m_0 c4\pi\varepsilon_0}\right]}_{B}\frac{1}{r}+\underbrace{\left[\left(\frac{(-Q)e}{m_0 c4\pi\varepsilon_0}\right)^2-\frac{8\pi G_N}{c^2}\frac{1}{2}\frac{Q^2}{4\pi\varepsilon_0}\right]}_{A}\frac{1}{r^2} \qquad (125)$$

Note that C is always positive, since the velocity must either vanish for $r\to\infty$ (maximum case with $v_{tot}=c$), or the particle cannot reach infinity (case with $v_{tot}<c$).

Thus we have

$$\dot{r}=\pm\sqrt{\frac{A}{r^2}+\frac{B}{r}-C}=\frac{dr}{d\tau}$$

$$\Leftrightarrow d\tau=\pm\frac{dr}{\sqrt{\frac{A}{r^2}+\frac{B}{r}-C}}=\pm\frac{rdr}{\sqrt{A+Br-Cr^2}}$$

$$=\pm\frac{(-1)}{2C}\frac{[(-2C)r+B-B]dr}{\sqrt{A+Br-Cr^2}}=\pm\frac{(-1)}{C}\frac{[\frac{d}{dr}(A+Br-Cr^2)-B]dr}{2\sqrt{A+Br-Cr^2}}$$

$$=\pm\frac{(-1)}{C}\frac{d}{dr}(\sqrt{A+Br-Cr^2})dr\pm\frac{B}{2C}\frac{dr}{\sqrt{A+Br-Cr^2}} \qquad (126)$$

In this expression it is

$$\frac{1}{\sqrt{A+Br-Cr^2}}=\frac{2\sqrt{C}}{\sqrt{B^2+4AC-B^2+4BCr-4C^2r^2}}$$

$$=\frac{(-1)}{\sqrt{C}}\frac{\sqrt{B^2+4AC}}{\sqrt{(B^2+4AC)-(B-2Cr)^2}}\frac{(-2C)}{\sqrt{B^2+4AC}}$$

$$=\frac{1}{\sqrt{C}}\frac{(-1)}{\sqrt{1-\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)^2}}\frac{d}{dr}\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)$$

$$=\frac{d}{dr}\left(\frac{1}{\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)\right),$$

Thus we have

$$d\tau=\pm\frac{(-1)}{C}\frac{d}{dr}(\sqrt{A+Br-Cr^2})dr\pm\frac{B}{2C}\frac{d}{dr}\left(\frac{1}{\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)\right)dr$$

$$\Rightarrow\tau-\tau_0=\pm\frac{(-1)}{C}\sqrt{A+Br-Cr^2}\pm\frac{B}{2C\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)$$

$$=\mp\frac{1}{2C\sqrt{C}}\sqrt{4AC+4BCr-4C^2r^2}\pm\frac{B}{2C\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)$$

$$=\mp\frac{1}{2C\sqrt{C}}\sqrt{4AC+B^2-B^2+2B2Cr-(2Cr)^2}\pm\frac{B}{2C\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)$$

$$=\mp\frac{1}{2C\sqrt{C}}\sqrt{B^2+4AC-(B-2Cr)^2}\pm\frac{B}{2C\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right)$$

$$=\mp\frac{\sqrt{B^2+4AC}}{2C\sqrt{C}}\sqrt{1-(\frac{B-2Cr}{\sqrt{B^2+4AC}})^2}\pm\frac{B}{2C\sqrt{C}}\arccos\left(\frac{B-2Cr}{\sqrt{B^2+4AC}}\right),$$

where the operation d/dr and the operation of multiplying with dr and summing up, which is called an integration, compensate each other. τ_0 is the initial constant in this summation process, i.e. a starting time. We set

$$\frac{B-2Cr}{\sqrt{B^2+4AC}}=:\cos(\psi) \qquad (129)$$

where ψ is a more convenient parameter for the trajectory.

$$\Leftrightarrow B-2Cr=\sqrt{B^2+4AC}\cos(\psi)\Leftrightarrow B-\sqrt{B^2+4AC}\cos(\psi)=2Cr$$

$$\Leftrightarrow r=\frac{B}{2C}\left(1-\frac{\sqrt{B^2+4AC}}{B}\cos(\psi)\right)$$

$$\Rightarrow \tau-\tau_0 = \mp\frac{\sqrt{B^2+4AC}}{2C\sqrt{C}}\sqrt{1-\cos^2(\psi)} \pm \frac{B}{2C\sqrt{C}}\arccos(\cos(\psi))$$
$$= \mp\frac{\sqrt{B^2+4AC}}{2C\sqrt{C}}\sin(\psi) \pm \frac{B}{2C\sqrt{C}}\psi$$
$$= \pm\frac{B}{2C\sqrt{C}}\left(\psi - \frac{\sqrt{B^2+4AC}}{B}\sin(\psi)\right)$$

$$\Leftrightarrow \pm\sqrt{C}(\tau-\tau_0) = \frac{B}{2C}\left(\psi - \frac{\sqrt{B^2+4AC}}{B}\sin(\psi)\right)$$

The equations (131) and (130) are the equations of an elongated cycloide for the coordinates ($\pm\sqrt{C}(\tau-\tau_0)$, r) = (x, y). τ_0 is a starting time, and the $\pm$ sign in equation (131) indicates that the cycloide can be crossed either in the positive or negative direction.

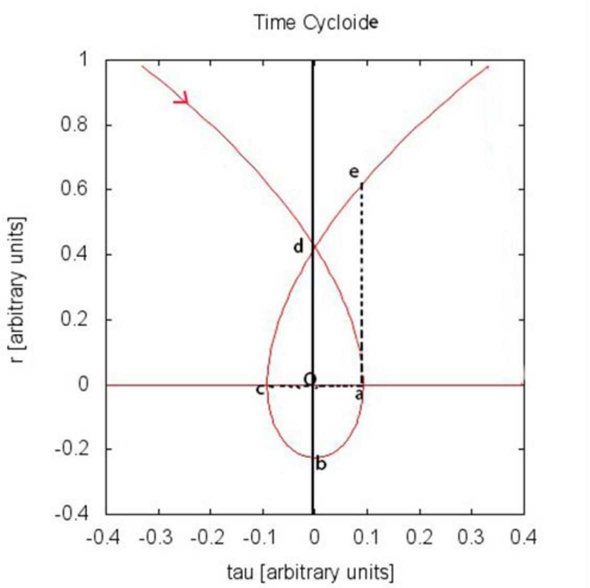

Figure 1: Cycloid of the space time spiral for parameters $A = 0.5$, $B = 2$, and $C = 1$. The object or spacecraft comes in from the left, crosses point d and reaches the gravitational center at point a. Then, the spacecraft travels backward in time, reaches the lowest (negative) point b, and is back at the height of the gravitational center at point c. From then on, the spacecraft travels forward in time again, reaching the point d and crossing point e.

Geometrically, the cycloid can be constructed in the following way: There is a point, which has a constant y coordinate $B/2C$, and which is shifted in the positive x direction by $B\psi/2C$, when the trajectory parameter ψ increases, and we look at the positive sign in equation (131). This moving point is the starting point of a rotating pointer, which has the length $B/(2C)\sqrt{B^2+4AC}/B$ and a rotation angle ψ, measured clockwise from the negative vertical. The end point of this pointer is the point on the cycloide, which belongs to the specific trajectory parameter ψ. We get the complete cycloid, when we let the whole thing rolling, i.e. increase ψ from zero to larger values.

In the graph of the cycloid, see Figure 1, r becomes zero at the point c when

$$\frac{\sqrt{B^2+4AC}}{B}\cos\psi_1 = 1 \Leftrightarrow \psi_1 = \arccos\left(\frac{B}{\sqrt{B^2+4AC}}\right) \quad (131)$$

The same point has a (negative) time coordinate

$$\sqrt{C}\tau_{0c} = \sqrt{C}(\tau(\psi_1)-\tau_0) = \frac{B}{2C}\left(\psi_1 - \frac{\sqrt{B^2+4AC}}{B}\sin\psi_1\right)$$

$$= \frac{B}{2C}\left(\psi_1 - \frac{\sqrt{B^2+4AC}}{B}\sqrt{1-\cos^2\left(\arccos\left(\frac{B}{\sqrt{B^2+4AC}}\right)\right)}\right)$$

$$= \frac{B}{2C}\left(\arccos\left(\frac{B}{\sqrt{B^2+4AC}}\right) - \frac{\sqrt{B^2+4AC}}{B}\sqrt{1-\frac{B^2}{B^2+4AC}}\right)$$

$$= \frac{B}{2C}\left(\arccos\left(\frac{B}{\sqrt{B^2+4AC}}\right) - \frac{\sqrt{B^2+4AC}}{B}\frac{\sqrt{B^2+4AC-B^2}}{\sqrt{B^2+4AC}}\right)$$

$$= \frac{B}{2C}\left(\arccos\left(\frac{B}{\sqrt{B^2+(2\sqrt{AC})^2}}\right) - \frac{2\sqrt{AC}}{B}\right)$$

Since the cosine of an angle is the ratio between the adjacent to the hypotenuse in a rectangular triangle, the angle $\arccos\left(\frac{B}{\sqrt{B^2+(2\sqrt{AC})^2}}\right)$ is the angle in the rectangular triangle with the adjacent B and the opposite $2\sqrt{AC}$. Thus, we have

$$\arccos\left(\frac{B}{\sqrt{B^2+(2\sqrt{AC})^2}}\right) = \arctan\left(\frac{2\sqrt{AC}}{B}\right),$$

since the tangent of an angle in a rectangular triangle is the ratio between the opposite and the adjacent.

$$\Rightarrow \sqrt{C}\tau_{0c} = -\frac{B}{2C}\left(\frac{2\sqrt{AC}}{B} - \arctan\left(\frac{2\sqrt{AC}}{B}\right)\right) \quad (134)$$

We are also interested in the point e in the diagram. It is the point above the time coordinate $-\sqrt{C}\tau_{0c}$, for which we have

$$\psi_2 - \frac{\sqrt{B^2+4AC}}{B}\sin\psi_2 = \frac{2\sqrt{AC}}{B} - \arctan\left(\frac{2\sqrt{AC}}{B}\right) \qquad (135)$$

and the estimate $\pi/2 < \psi_2 < \pi$. This equation has to be evaluated numerically. Yet, we can get an approximate solution, by substituting $\psi_2 = \pi/2 + \phi_2$, and noting that $\sin\psi_2 = \cos\phi_2 \approx 1 - \phi_2^2/2$. With these substitutions we get a quadratic equation for ϕ_2 from equation (135), which can be solved for ϕ_2. When we insert the result in equation(130), we get

$$r_{ae} \approx \frac{B}{2C}\left(1 - \frac{\sqrt{B^2+4AC}}{B}\cos\left[\left(\frac{\pi}{2} - \frac{B}{\sqrt{B^2+4AC}}\right) + \sqrt{\left(\frac{\pi}{2} - \frac{B}{\sqrt{B^2+4AC}}\right)^2 - \left(\frac{\pi^2}{4} - 2\right) + \frac{2B}{\sqrt{B^2+4AC}}\left[\frac{2\sqrt{AC}}{B} - \arctan\left(\frac{2\sqrt{AC}}{B}\right)\right]}\right]\right) \qquad (136)$$

for the distance between point a and point e.

Figure 1 is also the diagram for the navigation in space and time. We can read the diagram in two different ways:

A)i) The spacecraft comes in from the left, crosses point d, and reaches point a, where it plunges into the gravitational center. Since the spacecraft travels backward in time from then on, it has disappeared from our world, where we can only see objects traveling forward in time. However, at the same spatial level as the gravitational center, the spacecraft emerges at point c, from where on it propagates forward in time again. Yet, point c is at a time

$$\tau_{ca} = -2\tau_{0c} = \frac{B}{C\sqrt{C}}\left[\frac{2\sqrt{AC}}{B} - \arctan\left(\frac{2\sqrt{AC}}{B}\right)\right] \qquad (137)$$

earlier than point a (compare with equation (134)). This has the meaning that **the spacecraft has performed a jump τ_{ca} backward in time**.

A)ii) The spacecraft uses the negative branch of equation (131). Then, the space time spiral is crossed in the opposite direction. The spacecraft comes in from the right, plunges into the gravitational center at point c, and reemerges at a time $\tau_{ac} = -\tau_{ca}$ **earlier** at point a again. Thus, **the spacecraft has performed a jump τ_{ca} forward in time**.

The navigations of the spacecraft according to plan A)i) or A)ii) are **the principle of the time machine for traveling into the past or to the future**.

B) The spacecraft comes in from the left, crosses point d, and plunges into the gravitational center at point a. There it disappears. We can only observe what is traveling forward in time. Therefore, we observe that at the time of disappearance, **the spacecraft reappears at point e**, i.e. **the spacecraft has performed an instantaneous spacial jump r_{ae}** (compare with formula (136)). **This is the principle of instantaneous deep space travel**.

Note that in equation (136) there is a factor $1/C = 1/(c^2 - v_{tot}^2)$ on the right side. By choosing the initial speed v_{tot} of the spacecraft very close to the light velocity c, in which case the object can reach infinity, i.e. $1/C \to \infty$, we see that we can bridge any distance in space by instantaneous deep space travel. Thus, a traveler, who can set free arbitrary energies for the acceleration of his spacecraft, can reach any liked location in the universe instantaneously, no matter how far this location is.

In equation (137), for reasonable charges, we have $\sqrt{A} \approx \frac{(-Q)e}{m_0 c 4\pi\varepsilon_0}$, $B \approx -2v_{tot}\frac{(-Q)e}{m_0 c 4\pi\varepsilon_0}$, and we get the limits

$$\tau_{ca,v_{tot}\to c} \approx \frac{2}{3}\frac{1}{c^3}\left(\frac{(-Q)e}{m_0 4\pi\varepsilon_0}\right) \approx \frac{1}{3}\tau_{ca,v_{tot}\to 0}, \qquad (138)$$

where we used the polynomial approximation $\arctan(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots$ of the arctangent function for $|z| < 1$, and $\lim_{z\to\infty}\arctan(z) = \frac{\pi}{2}$. From equation (138) we can see that we can get arbitrarily large temporal jumps, if we increase the charge Q of the gravitational center, increase the charge e of the object, or decrease the mass m_0 of the object. On the other hand, we can see that the temporal jump is of the order $1/c^3$. Thus, it is a third order effect in the dynamics of micro physical systems.

As an example, we can now look at a electron - positron system, where a negative elementary particle, called electron, orbits a positive elementary particle, called proton, which charge has the same absolute value as the electron, and which has roughly 1836 times the mass of the electron. This system constitutes the most simple atom we can think of. For this system we get

$$\tau_{uncertain} = \hbar/(2m_0c^2) = 6.4405\times10^{-22}\,sec \approx 34.259\,\tau_{ca,v_{tot}\to 0},$$

where $\tau_{uncertain}$ is the uncertainty in time for a measurement of an electron in a process where the electron with total energy m_0c^2 gets annihilated and recreated again.

If we take the process of an electron repeatedly plunging in an neighboring proton as an stochastic process, i.e. successive dives of the electron are correlated up to a certain extend, then a certain number n that defines the maximal number of successively correlated dives exist. With n given,

$$b_{n,p}(k) = \binom{n}{k} p^k (1-p)^{n-k}, \qquad (139)$$

where p is the probability for a single dive, is the probability with which k successive dives occur, where $0 \le k \le n$. The probability distribution (139) over k is called "binomial distribution". For this binomial distribution the relationship

$$\sigma b_{n,p}(k) \approx \phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \ where \ x = \frac{k-np}{\sigma} \ and \ \sigma = \sqrt{np(1-p)}$$

holds [see, e.g., Reinhardt & Soeder, 1982]. The probability function $\phi(x)$ is called "normal distribution". The relationship (140) is the better fulfilled the larger n becomes and becomes exact for $n \to \infty$. However, one can look at an approximation accuracy function

$$S_{p,n} = \sum_{k=0}^{n} \left[\frac{1}{\sqrt{2\pi}} e^{-\frac{(k-np)^2}{2np(1-p)}} - \sqrt{np(1-p)} \binom{n}{k} p^k (1-p)^{n-k} \right]^2$$

between the normal and the binomial distribution, which is a function of n. Figure 2 shows $S_{p,n}$ for $p = 1/2$.

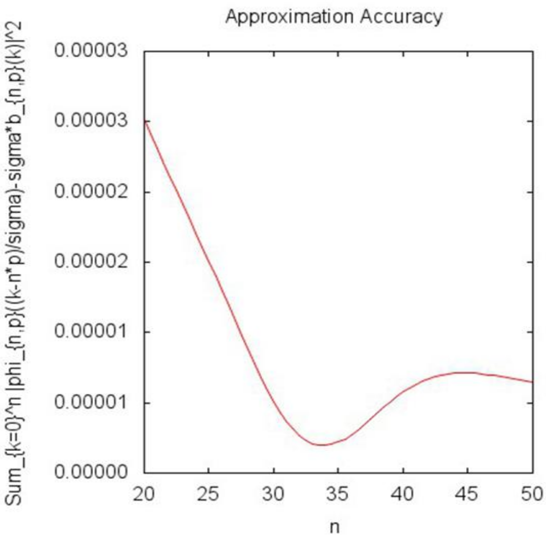

Figure 2: Approximation accuracy $S_{p,n}$ between the normal and the binomial distribution for $p = 1/2$.

One finds that $S_{p,n}$ is generally a monotonously falling function with n. Yet, $S_{p,n}$ has a slight irregularity in the vicinity of $n \approx 34$. At a local minimum $n_{min} = 34.259$ the binomial distribution is particularly close to the normal distribution. This means that in a stochastic process where about 34 elementary processes are correlated, the normal distribution is already a very good approximation to the probability distribution of the stochastic process. The normal distribution is again a very good approximation to the probability distribution of the stochastic process for much larger n. However, the more correlated elementary processes are involved, the less likely the corresponding stochastic process is realized in nature. Thus, one indeed finds that in most stochastic processes in nature about 34 elementary processes are correlated that can be described using a normal distribution as an empirical law. (The communication of that law is the merit of Dagmar Richter, who was my high school teacher for mathematics and physics, who had an education in statistical physics, and who is the widow of Arne Richter, the long-term general secretary of the European Geophysical Union (EGU). I would like to introduce here the constant $C_{Richter} = n_{min} = 34.259$ as the "Richter constant".) (140)

Thus, in a natural stochastic process that can be described with a normal distribution, the expectation value for the change of the stochastic variable τ is

$$\mathrm{E}(\tau) = C_{Richter} \tau_{elem}, \tag{142}$$

where τ_{elem} is the change of the stochastic variable τ in the elementary stochastic process.

Hence, we have

$$\mathrm{E}(\tau) = C_{Richter} \tau_{ca,v_{tot} \to 0} = \hbar/(2m_0c^2) = \tau_{uncertain} \quad (141)$$

for the expectation value of a temporal displacement of an electron for the stochastic process of temporal displacements of electrons successively plunging into neighboring protons, which is exactly the uncertainty time according to Heisenberg's uncertainty principle for a process in which an electron is annihilated and recreated, i.e. enters and exits a charged gravitational center.

We can rewrite (143) as

$$\hbar = 2m_0c^2 C_{Richter} \tau_{ca,v_{tot} \to 0}, \tag{144}$$

where $\tau_{ca,v_{tot} \to 0}$ is taken from (137) for $v_{tot} \to 0$, which means that we have found a derivation for the empirical Planck constant $h = 2\pi\hbar$.

Alternatively, we can write

$$[C_{Richter} \tau_{ca,v_{tot} \to 0}][m_0c^2] = \frac{\hbar}{2}$$

$$\Leftrightarrow \ \Delta t \Delta E = \frac{\hbar}{2}, \tag{145}$$

for the time and energy variations Δt and ΔE of the electron in the electron-proton system, which is Heisenberg's uncertainty principle for time and energy.

If one considers that a disturbance of the system would propagate with the light velocity, one gets

$$\Delta x = c\Delta t \ \ and \ \ \Delta p = \frac{1}{c} \Delta E \tag{146}$$

for the spatial variations Δx and the variations in the momentum Δp, and hence

$$\Delta x \Delta p = \frac{\hbar}{2}, \tag{147}$$

which is Heisenberg's uncertainty principle for spatial and momentum variations. Empirically, one finds that one has to set

$$\Delta t \Delta E \geq \frac{\hbar}{2}$$

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

if the probability functions do not describe idealistic states, which increases the uncertainties.

Mathematically,

$$\Delta x = \sqrt{\int \psi^* (\underline{x} - \int \psi^* \underline{x} \psi)^2 \psi}, \tag{150}$$

where ψ is the probability distribution or probability amplitude of the particle, ψ^* is the complex conjugate of that quantity, $\underline{x}$ is the spatial position vector, and the integrals are carried out over the entire spatial domain, and

$$\Delta p = \sqrt{\int \psi^* (-i\hbar\nabla - \int \psi^* (-i)\hbar\nabla\psi)^2 \psi}, \qquad (151)$$

since the operator $-i\hbar\nabla$ retrieves the momentum from a wave function describing a photon field (which carries rotational momenta in multiples of h).

It is a straightforward calculation that the probability distribution

$$\psi = \frac{1}{\sqrt{\pi}} \bar{\gamma}^{3/2} \frac{e^{-\bar{\gamma}r}}{r}, \qquad (152)$$

where $\bar{\gamma} = \frac{m_0 e^2}{\hbar^2} = 1/a_0$ is the reciprocal so-called "Bohr radius" a_0, satisfies (147) exactly with (150) and (151) [see, e.g., Greiner, 1984]. (Consider that in this notation of the atomic physicists the parameter e^2 for the product of the charges contains the Coulomb constant.) Note that (152) becomes maximal when $r \to 0$, and hence describes our stochastic process of electrons plunging into the gravitational center of the proton quite well.

On the other hand, (152) satisfies

$$-\frac{m_0 e^4}{2\hbar^2}\psi = -\frac{\hbar^2}{2m_0}\nabla^2\psi - \frac{e^2}{r}\psi, \qquad (153)$$

where $E_1 = -\frac{m_0 e^4}{2\hbar^2} = -\frac{1}{2}\frac{e^2}{a_0}$ is the energy of an electron orbiting the proton on a classical circular trajectory with the radius a_0.

From here, it is straightforward to postulate that any other resonant state in the hydrogen atom must satisfy an equation

$$E\psi = -\frac{\hbar^2}{2m_0}\nabla^2\psi - \frac{e^2}{r}\psi \qquad (154)$$

for energy values E_n for E for which normalizable solutions ψ exist. This then generalizes to

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m_0}\nabla^2\psi - V(\underline{x})\psi \qquad (155)$$

in the non-stationary case and a general interaction potential $V(\underline{x})$, since the operation $i\hbar\partial_t$ retrieves the energy from a wave function that describes a photon field that carries the energy in packages $\hbar\omega$, where $\omega = 2\pi f$ is the angular frequency of the wave.

Equation (155) is the so-called "Schroedinger equation", and with it we have derived quantum mechanics from the principle of electrons repeatedly plunging into neighboring protons.

CONCLUSIONS

VI. We have revisited the Kaluza-Klein theory and solved the field equations of the Kaluza-Klein theory with constant coupling field between the electromagnetic and gravitational field in terms of power expansions in the coordinate for the spherical symmetric case entirely. In the Einstein-Maxwell case where the electromagnetic field and the gravitational fields are coupled linearly, we discussed the exact behavior of the roots of the pseudo potential for the motion of the position as a function of the planar - angle for an orbiting particle. We investigated the analytic continuation of a trajectory of a test particle entering the gravitational center of a central body, which has performed a temporal jump when exiting the gravitational center again. This temporal displacement, if repeated, constitutes a stochastic process that has an expectation value of the reduced Planck constant divided by two times the rest mass of the electron, since the temporal displacement process of the electron goes along with an annihilation and recreation process of the electron that enters and exits the gravitational center. Thus, our finding corresponds to the existence of a Heisenberg uncertainty relation with respect to temporal and energetic fluctuations of the electron in the electron-proton system, which translates to an Heisenberg uncertainty relation with respect to spatial variations and variations in the momentum of the electron in the electron-proton system. The validity of the latter uncertainty relation is equivalent with the existence of a Schroedinger equation governing the statistic behavior of the electron in the electron-proton system. In this way we have derived the ground principles of classical quantum mechanics from the unified gravitational theory for gravitation and electromagnetism straightforwardly. Further work that investigates the behavior of test particles with non-zero rotational momentum and that orbit a charged gravitational center is under way.

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