International Journal of Technical Research and Applications e-ISSN: 2320-8163,
www.ijtra.com Volume 3, Issue 2 (Mar-Apr 2015), PP. 123-125

A STUDY ON MARKOV CHAIN WITH TRANSITION DIAGRAM

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***Abstract*—Stochastic processes have many useful applications and are taught in several university programmers. In this paper we are using stochastic process with complex concept on Markov chains which uses a transition matrix to plot a transition diagram and there are several examples which explains various type of transition diagram. The concept behind this topic is simple and easy to understand.**

***Keywords*— Transition Diagram, Transition Matrix, Markov Chain, First Passage Time, Persistent State, Transient State, Periodic State, Inter-Communicating States.**

I. INTRODUCTION

A Stochastic processes are important for modelling many natural and social phenomena and have useful applications in computer science, physics, biology, economics and finance. There are many textbooks on stochastic processes, from introductory to advanced ones [2] & [4]. Discrete Markov processes are the simplest and most important class of stochastic processes.

There are only few publications on teaching stochastic processes and Markov chains, such as [1] & [5] "More research needs to be carried out on how to teach stochastic processes to researchers" [5].

"While trying to understand Markov chains models, students usually encounter many obstacles and difficulties" [6]. Many lecturers use visual displays such as sample paths and transition diagrams to illustrate Markov chains. In this article we utilise transition diagrams further for teaching several important concepts of Markov chains. We explain in details how these concepts can be defined in terms of transition diagrams (treated as directed weighted graphs) and we accompany this with worked examples. Transition diagrams provide a good techniques for solving some problems about Markov chains, especially for students with poor mathematical background.

II. TRANSITION DIAGRAM OF A MARKOV CHAIN: DEFINITIONS

A homogeneous finite Markov chain is entirely defined by its initial state distribution and its transition matrix S = $[p_{ij}]$, where $p_{ij} = P(X_1 = i|X_0 = j)$ is the transition probability from state j to state i.

The graphical representation of a Markov chain is a transition diagram, which is equivalent to its transition matrix.

The transition diagram of a Markov chain X is a single weighted directed graph, where each vertex represents a state of the Markov chain and there is a directed edge from vertex j to vertex i if the transition probability p_{ij} >0; this edge has the weight/probability of p_{ij}.

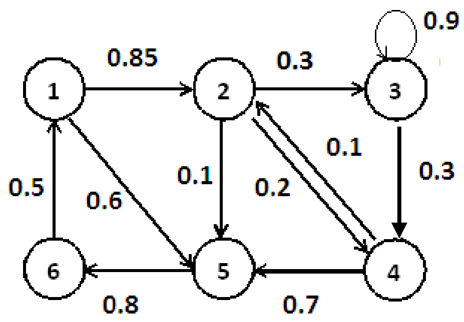


Fig.1.The transition diagram of the Markov chain Example1.

Example 1

A Markov chain has states 1, 2, 3, 4, 5, 6 and the following transition matrix:

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0.5 \\ 0.85 & 0 & 0 & 0.1 & 0 & 0 \\ 0 & 0.3 & 0.9 & 0 & 0 & 0 \\ 0 & 0.2 & 0.3 & 0 & 0 & 0 \\ 0.6 & 0.1 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.8 & 0 \end{bmatrix}$$

This is its transition diagram.

In the diagram in **Fig. 1** the probability of each edge is shown next to it. For example, the loop from state 3 to state 3 has probability $0.9 = p_{33} = P(X_1 = 3 \mid X_0 = 3)$ and the edge from state 2 to state 3 has probability $0.3 = p_{32} = = P(X_1 = 3 \mid X_0 = 2)$.

In the graph terminology, an edge sequence of length n is an ordered sequence of edges $e_1, e_2, \ldots, e_n$, where e_i and e_{i+1} are adjacent edges for all i = 1, 2,…, n-1.

A path is an edge sequence, where all edges are distinct. A *simple path* is a path, where all vertices are distinct (except possibly the start and end vertices). A *cycle* is a simple path there the start vertex and the end vertex are the same.

In a transition diagram the probability of an edge sequence equals a product of the probabilities of its edges.

III. PROPERTIES OF A MARKOV CHAIN IN TERMS OF TRANSITION DIAGRAMS

A. N-Step Transition Probability:

An n-step transition probability is:

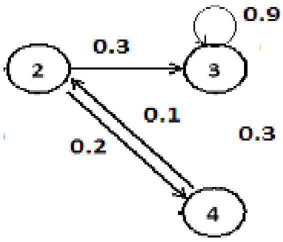
$$p_{ij}^{(n)} = P (X_n = i \mid X_0 = j) .$$

It equals the probability of getting from state j to state i in exactly n steps. It can be calculated as the corresponding element of the matrix $S^{(n)}$ but it is usually easier to find it from the transition diagram as a sum of the probabilities of all edge sequences of length n from j to i.

Example 2

In the chain from Example 1, the 3-step transition probability from 2 to 1 equals:

$$p_{32}^{(3)} = a_1 + a_2$$



Where:

a_1 = The probability of the path 2423
$= 0.2 \times 0.1 \times 0.3 = 0.006$
a_2 = The probability of the edge sequence 2333.
$= 0.3 \times 0.9^2 = 0.243$

These probabilities are easy to find from the diagram
$p_{32}^{(3)} = 0.006 + 0.243 = 0.249.$
$p_{32}^{(3)} = 0.249.$

B. Probability of Visiting a State for the First Time:

Let us consider a random variable:

$T_i = \min\{n \geq 1 : X_n = i\}$.

It represents the number of steps to visit a state i for the first time. It is called the first passage time of the state i. Related probabilities are:

$f_{ij}^{(m)} = P(T_i = m \mid X_0 = j)$ and $f_{ij} = P(T_i < \infty \mid X_0 = j)$.

Clearly:

$$f_{ij} = \sum_{m=1}^{\infty} f_{ij}^{(m)}$$

These probabilities can be interpreted as follows:

$f_{ij}^{(m)}$ = the probability to visit i on step m for the first time starting from j;

$f_{ij}^{(m)}$ = the probability to visit i in finite number of starting from j;

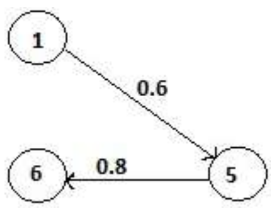
In terms of transition diagrams, f_{ij} equals a sum of the probabilities of all edge sequences from j to i that do not include the vertex i between the start and end vertices.

$f_{ij}^{(m)}$ equals a similar sum for the edge sequences of length m only.

For finite Markov chains these probabilities are easier to find from their transition diagrams than with other methods.

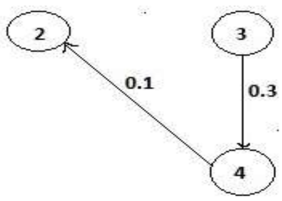
Example.3

From the transition diagram

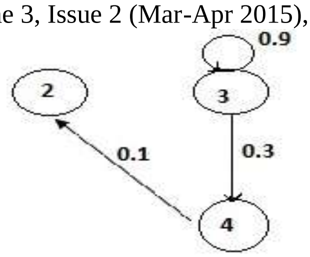


we can calculate the following probabilities:

- $f_{61}^{(2)} = 0.8 \times 0.6 = 0.48$ as the probability of the path 156.
  $f_{61}^{(n)} = 0$ for $n \neq 2$, $f_{61} = 1$
- For vertices 2 and 3 we have
  $f_{23}^{(1)} = 0$
  $f_{23}^{(2)} = 0.3 \times 0.1 = 0.03$ as the probability of the path 342.



$f_{23}^{(3)} = 0.9 \times 0.3 \times 0.1 = 0.027$ as the probability of the path 3342.



and in general, for $n \geq 0$, for $f_{23}^{(n+1)} = (0.9)^n \times 0.3 \times 0.1$ as the probability of the edge sequence $\underbrace{3\ldots3}_{(n+1)times}\ 42$ with n loops around 3.

So:

$f_{23} = \sum_{m=1}^{\infty} f_{23}^{(n)} = \sum_{n=0}^{\infty} 0.9^n \times 0.3 \times 0.1$
$= 0.3 \times 0.1 \times \frac{1}{1-0.9} = 0.3$
$f_{23} = 0.3.$

C. Persistent and Transient States

A state i of a Markov chain is called *persistent* if $f_{ii} = 1$ and *transient* otherwise.

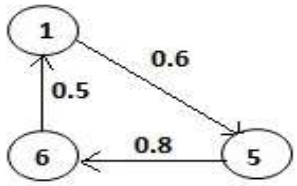
Thus, if the chain starts at a persistent state, it will return to this state almost surely. If the chain starts at a transient state, there is a positive probability of never returning to this state. From the transition diagram we can evaluate the probability f_{ii} and therefore determine whether the state i is persistent or transient.

Example 4

For each of the states 3 and 6 of the Markov chain in Example 1 determine whether the state is persistent or transient.

Solution

- $f_{66} = f_{66}^{(3)} = 0.24$ as the probability of the cycle 615. So the state 6 is persistent.



- $f_{33}^{(1)} = 0.9$ the probability of the loop around 3
  $f_{33}^{(2)} = 0$
  $f_{33}^{(3)} = 0.3 \times 0.1 \times 0.3 = 0.009$ as the probability of the cycle 3423.

More generally, for any $n \geq 1$, $f_{33}^{(2n)} = 0$ and $f_{33}^{(2n+1)} = 0.1 \times 0.3^n$ as the probability of the edge sequence $3\underbrace{42\ldots42}_{ntimes}3$.

So:

$f_{33} = \sum_{m=1}^{\infty} f_{33}^{(m)} = 0.9 + 0.3 \times 0.1 \sum_{n=1}^{\infty} 0.3^n$
$= 0.9 + 0.3 \times 0.1 \times \frac{0.3}{1-0.3} = 0.9$

Since $f_{11} = 0.9 < 1$, the state 3 is transistent.

Lemma 1

Suppose i and j are two different states of a Markov chain. If $p_{ji} > 0$ and $f_{ij} = 0$, then the state i is transient. This lemma is easily derived from the definition of f_{ij}. The lemma can be rephrased in terms of transition diagrams: if the chain can reach state j from state i in one step ($p_{ji} > 0$) but cannot come back ($f_{ij} = 0$), then the state i is transient.

Lemma 1 gives a method of finding transient states from a transition diagram without any calculations. For example, from **Fig. 1** we can see that $p_{16} = 0.3 > 0$ and $f_{61} = 0$ because the chain cannot return from state 6 to state 1. Therefore by Lemma 1 the state 1 is transient. This is consistent with the result of Example 4.

D. Mean Recurrence Time:

The *mean recurrence time* of a persistent state i is

defined as $$\mu_i = \sum_{m=1}^{\infty} m f_{ii}^{(m)}.$$

If i is a transient state, $\mu_i = \infty$ by the definition.

Thus μ_i is the expected time of returning to the state i if thechain states at i.

Example 5

For each of the states 3 and 6 of the Markov chain in Example 1 find its mean recurrence time.

Solution

- Since the state 3 is transient $\mu_3=\infty$.
- For the state 6, $f_{66}^{(3)}$=0.24 and $f_{66}^{(n)}$=0 for any $n\neq 3$

So $\mu_6=3\times f_{66}^{(3)}=3\times 0.24=0.72$.

E. Periodic States:

The period of a state i is the greatest common divisor of all $n \geq 1$ with $p_{ii}^{(n)} > 0$.

The state i is *periodic* if its period is greater than 1; otherwise it is aperiodic.

In terms of transition diagrams, a state i has a period d if every edge sequence from i to i has the length, which is a multiple of d.

Example 6

For each of the states 3 and 6 of the Markov chain in Example 1 find its period and determine whether the state is periodic.

Solution

The transition diagram in **Fig. 1** has a cycle 343 of length 2 and a cycle 3432 of length 3. The greatest common divisor of 2 and 3 equals 1. Therefore the period of the state 2 equals 1 and the state is aperiodic.

Any edge sequence from 6 to 6 is a cycle 615 or its repetition, so its length is a multiple of 3. Hence the state 6 is periodic with period 3.

F. Communicating States

State i *communicates* with state j (notation $i\rightarrow j$) if $p_{ji}^{(n)} > 0$ for some $n \geq 0$.

In terms of transition diagrams, a state i communicates with state j if there is a path from i to j.

State i *inter-communicates* with state j (notation $i\leftrightarrow j$) if the two states communicate with each other.

Theorem 1

(Grimmett and Stirzaker, 2001)

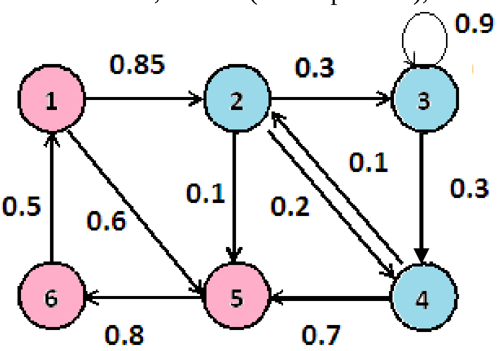
Suppose i and j are two states of a Markov chain and $i\leftrightarrow j$. Then:

- i and j have the same period;
- i is persistent $\Leftrightarrow$ j is persistent;
- i is transient $\Leftrightarrow$ j is transient

Inter-communication is an equivalence relation on the set Q of all states of a Markov chain. So the set Q can be partitioned into equivalence classes;all states in one equivalence class share the same properties, according to Theorem 1.

Example 7

Let us consider the Markov chain from Example 1 and its transition diagram



Clearly, the states 2 and 4 inter-communicate. Also $2\rightarrow 3$, since $p_{32} > 0$ and $3\rightarrow 2$, since there is a path 342 from 3 to 2.

Next, $6\rightarrow 1$ but not $1\rightarrow 6$ (there is no path from 6 to 1). States 1, 5 and 6 all inter-communicate.

Therefore, the equivalence class of 3 is:

[3] = {2,3,4}

and the equivalence class of 6 is:

[6] = {1,5,6}.

According to Theorem 1 and Examples 4 and 5, the states 2, 3 and 4 are all transient and aperiodic; the states 1, 5 and 6 are all persistent and periodic with period 3.

IV. Conclusion

In this paper we have given some elementary definitions for Markov chains. These definitions are utilized in plotting the transition diagrams. Finally we have related the two components that are Markov chains and transition matrix in the form of diagram.

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