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MHD FULLY DEVELOPED FLOW OF A THIRD GRADE FLUID IN A PLANE DUCT

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Abstract— **In this paper, an appropriate analysis has been performed to study the incompressible fully developed flow of a non-Newtonian third grade fluid in a plane duct under an externally applied magnetic field. The governing equations, continuity, momentum and Ohm's law for this problem are reduced to an ordinary form and are solved by Homotopy Analysis Method (HAM). The present study works on new algorithm which proposes more suitable initial function with faster convergence to final solution in comparison with traditional method in HAM. From the physical point of view, the results indicate that the behavior of non¬-Newtonian third grade fluid flow approached the Newtonian one with increasing the magnetic field strength.**

Index terms— **Fully developed, Magnetohydrodynamic, Plane duct, Third grade**

I. INTRODUCTION

The magnetohyrodynamcs (MHD) phenomenon is characterized by an interaction between the hydrodynamic boundary layer and the electromagnetic field. Recently there has been an increasing interest in fluid flow through MHD channel because many applications of them are being used in engineering. An extensive theoretical work has been carried out on the hydromagnetic fluid flow in a channel under various situations by Hartmann [1]. Theoretical investigation of the applicability of magnetic fields for controlling hydrodynamic separation in Jeffrey-Hamel flows of viscoelastic fluids has been studied by Sadeghy et al. [2]. The MHD Flow of Compressible Fluid in a Channel with Porous Walls is Investigated by Pourmahmoud et al. [3]. Although most of the common fluids in the real world exhibit Newtonian behavior, there are important classes of fluids that are classified as non-Newtonian. Non-Newtonian fluids are those, whose constitutive equation, the equation that relates the stress and strain, is not a simple linear relation. Due to non-linear dependence, the analysis of the behavior of non-Newtonian fluids presents exciting challenges to Engineers and mathematicians. There is not a single constitutive equation which can describe the flow behavior of all the non-Newtonian fluids. Because of the complex microstructure of fluids, various models have been proposed to predict the non-Newtonian behavior. Several investigators are now engaged in getting the solutions under different physical aspects. One of special cases among these classes of fluids which can be solved analytically is the second grade fluid. Baris et al. [4], Mohyuddin et al. [5], Ali et al.[6], and Chauhan et al. [7] studied second grade fluid in channel at various situations. Although the second grade fluid model for steady flow is able to show the normal stress effects, it does not take into account the shear thinning or shear thickening phenomena that many fluids show. The third grade fluid model represents a further attempt toward a more comprehensive description of the behavior of non-Newtonian fluids. The third grade fluid in channel is studied by Roohi et al. [8] and Mohyuddin et al. [9] in different situations. Keeping this importance of third grade fluid, our concern in this paper is to investigate the effect of magnetic strength on channel caring third grade fluid, in fully developed region. The governing differential equation is nonlinear and second order. The mentioned equation is solved by applying the homotopy analysis method (HAM). It provides an efficient explicit solution with minimal calculations. The HAM was first proposed by Liao in 1992 and then was developed by him [10-13]. This method has been successfully applied to solve many types of nonlinear problems [14-18]. In HAM solutions, we should choose an initial guess function, u_0, and auxiliary linear operator, L. If the number of boundary conditions and L order increased, the u_0 will have better convergence to final solution, i.e. $u(y)$. For this purpose differentiate governing equation and the result is used as main equation. The first equation is used as additional boundary condition. The accuracy of HAM is authenticated by comparing with numerical results.

A. Problem statement

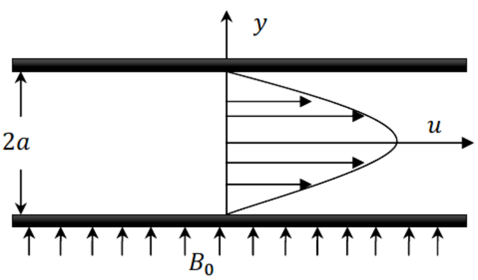

Fig.1. A sketch of the physical problem

Let us consider the fully developed laminar flow of an incompressible and electrically conducting fluid in a channel as shown in Fig.1. The no slip boundary conditions are exerted on walls. The uniform magnetic field, B_0, is imposed along the y-axis. The governing equations, continuity, momentum and Ohm's law for the problem can be written as follows:

$$\nabla . V = 0, \tag{1}$$

$$\rho \frac{dV}{dt} = -\nabla p + divS + J \times B, \tag{2}$$

$$J = s\left(E + V \times B\right), \tag{3}$$

Where ρ, V, p, S, σ and J are the density, velocity vector, pressure, extra stress tensor, electrical conductivity and current density respectively. The d/dt denotes the material time derivative, $B = B_0 + b$ (b being the induced magnetic field and B_0 an external magnetic field), is the total magnetic field and E is the electric field. It is assumed that the magnetic Reynolds Number is small and the induced magnetic field, b, due to the motion of the electrically conducting fluid is negligible. It is also assumed that the electrical conductivity of fluid, σ, is constant and the external electric field is zero. Under these assumptions the last term in Eq. (2), The Lorentz force per unit volume is given by:

$$J \times B = -s B_0^2 u. \tag{4}$$

The extra stress tensor S is defined as [8-9]

$$S = \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2 + \beta_1 A_3 + \beta_2\left(A_1 A_2 + A_2 A_1\right) + \beta_3\left(trA_1^2\right) A_1, \tag{5}$$

μ being the coefficient of shear viscosity $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3$ are material constants. The tensors A_1, A_2, A_3 are given by:

$$A_1 = \left(gradV\right) + \left(gradV\right)^T,$$
$$A_2 = \frac{d}{dt} A_1 + A_1\left(gradV\right) + \left(gradV\right)^T A_1, \tag{6}$$
$$A_3 = \frac{d}{dt} A_2 + A_2\left(gradV\right) + \left(gradV\right)^T A_2.$$

The flow is fully-developed, then the velocity and extra stress are dependent of y only, then;

$$V = \left[u(y), 0, 0\right], \tag{7}$$

$$S = S\left(y\right). \tag{8}$$

Under these assumptions and definitions, the velocity field automatically satisfies the continuity equation and the momentum equations can be written as follow:

$$\frac{dS_{xy}}{dy} - \frac{dp}{dx} - \sigma B_0^2 u = 0, \tag{9}$$

$$\frac{dp}{dy} = 0. \tag{10}$$

For the fully developed flow the pressure gradient is constant and then Eq. (9) can be written as:

$$\frac{dS_{xy}}{dy} - \sigma B_0^2 u = \frac{dp}{dx} = cons. \tag{11}$$

By using Eqs. (5-8), the expression for the stress is;

$$S_{xy} = \mu \frac{du}{dy} + 2\beta\left(\frac{du}{dy}\right)^3, \quad \beta = \beta_2 + \beta_3. \tag{12}$$

By using Eq. (11) and Eq. (12), the momentum equation is:

$$\mu \frac{d^2u}{dy^2} + 6\beta\left(\frac{du}{dy}\right)^2 \frac{d^2u}{dy^2} - \sigma B_0^2 u = \frac{dp}{dx}. \tag{13}$$

The flow is symmetric about the center line of the channel, $y = 0$, and we only focus our attention on the flow in the region $0 \le y \le a$ .The boundary conditions for this problem can be written as:

$$\frac{\partial u}{\partial y} = 0 \quad at \quad y = 0, \ \left(i.e. symmetry\right), \qquad u = 0 \ \ at \ \ y = a. \tag{14}$$

The following dimensionless variables are introduced:

$$y^* = \frac{y}{a}, \qquad u^* = -\frac{\mu u}{a^2(\partial p/\partial x)}$$
$$T = \frac{6\beta a^2(\partial p/\partial x)^2}{\mu^3}, M^2 = \frac{\sigma B_0^2 a^2}{\mu}. \tag{15}$$

Where T and M are the dimensionless non-Newtonian coefficient and Hartmann number respectively. By substituting these changed variables, which were introduced in Eq. (15), into Eq. (13) and Eq. (14) we obtain:

$$\frac{d^2u^*}{dy^{*2}} + T\left(\frac{du^*}{dy^*}\right)^2 \frac{d^2u^*}{dy^{*2}} - M^2 u^* + 1 = 0, \tag{16}$$

$$\frac{\partial u^*}{\partial y^*} = 0 \quad at \quad y^* = 0, \qquad u^* = 0 \ \ at \ \ y^* = 1. \tag{17}$$

In HAM solutions, we should choose the auxiliary linear operator, L. If the number of boundary conditions and order of L increased, the guess function, u_0, which is obtained by solving the differential equation corresponding to L, can satisfy appropriately the final solution, $u(y)$. For this purpose differentiate Eq. (16) ends to following result, (neglecting the star for clarity):

$$\frac{d^3u}{dy^3}+T[2(\frac{du}{dy})(\frac{d^2u}{dy^2})^2 +(\frac{du}{dy})^2(\frac{d^3u}{dy^3})]-M^2\frac{du}{dy}=0\ . \tag{18}$$

By Eq.(16), introduce additional boundary condition as:

$$\frac{d^2u}{dy^2}+T\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}-M^2u+1=0 \quad at\ \ y=0\ . \tag{19}$$

B. Analytical solutions for $u(y)$

For HAM solutions, we choose auxiliary linear operator and corresponding differential equation in the following form:

$$L(u)=u''' , \tag{20}$$

$$u'''=0 , \tag{21}$$

The guess function is obtained by solving Eq.(21) with boundary conditions mentioned in Eq.(17) and Eq.(19) as below,

$$u_0(\mathrm{y})=\frac{1}{2+M^2}(1-y^2) , \tag{22}$$

Let $P\in[0,\ 1]$ denotes the embedding parameter and $\hbar$ indicates non–zero auxiliary parameters. Then we construct the following equations [10-13]. The zeroth- order deformation equation and corresponding boundary conditions are:

$$(1-P)L[U(y,p)-u_0(y)] = p\hbar N[U(y,p)], \tag{23}$$

$$U(1;p)=0,\ \ U'(0;p)=0, \quad U''(0;p)-M^2U(0;p) +T(U'(0;p))^2U''(0;p)=-1 \tag{24}$$

The non-linear differential operator $N[U(y,p)]$ is constructed by using Eq.(18) as below:

$$N[U(y,p)]=\frac{d^3U(y,p)}{dy^3}-M^2\frac{dU(y,p)}{dy} +2T\frac{dU(y,p)}{dy}\left(\frac{d^2U(y,p)}{dy^2}\right)^2 +T\left(\frac{dU(y;p)}{dy}\right)^2\frac{d^3U(y,p)}{dy^3}. \tag{25}$$

For $p=0$ and $p=1$ we have:

$$U(y,0)=u_0(y)\ ,\ U(y,1)=u(y)\ . \tag{26}$$

When p increases from 0 to 1 then $U(y,p)$ varies from $u_0(y)$ to $u(y)$. By Taylor's theorem and using Eq. (26), $U(y,p)$ can be expanded in a power series of p as follows:

$$U(y,p)=u_0(y)+\sum_{m=1}^{\infty}u_m(y)p^m\ , \quad u_m(y)=\frac{1}{m!}\frac{\partial^m(U(y,p))}{\partial p^m}\bigg|_{p=0}\ , \tag{27}$$

In which $\hbar$ is chosen in such a way, that this series converges to $u(y)$ at $p=1$, therefore we have :

$$u(y)=u_0(y)+\sum_{m=1}^{\infty}u_m(y)\ . \tag{28}$$

Differentiating m times the zeroth order deformation Eq. (23) with respect to p, then dividing by $m!$ and substitute $p=0$, we have the mth-order deformation equation as below[13]:

$$L[u_m(y)-\chi_m u_{m-1}(y)]=\hbar R_m(y), \tag{29}$$

In which

$$R_m(\mathrm{y})=u'''_{m-1}-M^2u'_{m-1} +\sum_{k=0}^{m-1}\left[\begin{array}{l}2T\times u''_{m-1-k}\left(\sum_{l=0}^{k}(u''_{k-l}\ \ u'_l)\right)\\ +T\times u'_{m-1-k}\left(\sum_{l=0}^{k}(u'_{k-l}\ \ u'''_l)\right)\end{array}\right], \tag{30}$$

And

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}, \tag{31}$$

With boundary conditions:

$$u_m(1)=0,\quad u'_m(0)=0, \quad u''_m(0)+T(u'_m(0))^2u''_m(0) -M^2u_m(0)=0. \tag{32}$$

Now, we have found the answer by maple software. The first-order deformation equation and corresponding boundary conditions are presented below:

$$u_1'''(y)=\hbar\,[u_0''''(y)+2T\,u_0'(y)\,(u_0''(y))^2 + T\,(u_0'(y))^2\,u_0'''(y)-M^2 u_0'(y)\,], \quad (33)$$

$$u_1(1)=0, \quad u_1'(0)=0, \quad u_1''(0)+T\,(u_1'(0))^2\,u_1''(0)-M^2 u_1(0)=0\,. \quad (34)$$

Solving the above equations subsequently results the following answers:

$$u_1(y)=-\frac{1}{18}\,\frac{\hbar\,(-4T+12M^2+12M^4+3M^6)}{32M^2+24M^4+8M^6+M^8+16} - \frac{1}{36}\,\frac{\hbar M^2(-4T+12M^2+12M^4+3M^6)}{32M^2+24M^4+8M^6+M^8+16}\,y^2 + \frac{1}{12}\,\frac{\hbar(-\frac{4}{3}T+4M^2+4M^4+M^6)}{(2+M^2)^3}\,y^4\,. \quad (35)$$

For higher order, the solutions were too long to be mentioned here. For 20th-order approximation the solutions converge for all selected values in $M \geq 0$, $0 \leq T \leq 6$ when ħ=-0.14, as will be shown in section 4. For $(m = 1,2,\ldots,20)$, u_m is obtained by solving Eq.(29) with boundary conditions which described in Eq.(32). Then, we have the final solution as:

$$u(y)=u_0(y)+\sum_{m=1}^{20} u_m(y)\,. \quad (36)$$

For example, if $(M = 3, T = 6)$ we have following expression,

$$\begin{aligned} u(y) &= 0.099239-0.05342\,y^2 \\ &-0.039622y^4-8.7495\times10^{-3}y^6 \\ &+1.3474\times10^{-3}y^8+1.2274\times10^{-3}y^{10} \\ &+1.1183\times10^{-4}y^{12}-1.0641\times10^{-4}y^{14} \\ &-2.638\times10^{-5}y^{16}+3.3835\times10^{-6}y^{18} \\ &+1.6923\times10^{-6}y^{20}+4.0603\times10^{-8}y^{22} \\ &-4.2411\times10^{-8}y^{24}-3.7487\times10^{-9}y^{26} \\ &+3.7150\times10^{-10}y^{28}+5.4005\times10^{-11}y^{30} \\ &-3.0735\times10^{-13}y^{32}-2.3484\times10^{-13}y^{34} \\ &-4.7259\times10^{-15}y^{36}+2.4305\times10^{-16}y^{38} \\ &+5.5480\times10^{-18}y^{40}-1.8088\times10^{-20}y^{42} \end{aligned} \quad (37)$$

For simplifying, let us consider $u(y)$ in the following from

$$u(y)=\sum_{i=0}^{20} g_i(y)\,. \quad (38)$$

We define the average values of functions $g_i(y)$ in interval $[0,1]$ as

$$\bar{g}_i=\int_0^1 g_i(y)\,dy\,. \quad (39)$$

For $(i=0\ \ to\ 42)$, let $\bar{g}_{i\max}$ be maximum magnitude of values $\bar{g}_i$ and the order of magnitude for $g_i(y)$ is defined as

$$OR_i=\frac{\bar{g}_i}{\bar{g}_{i\max}}\,. \quad (40)$$

By neglecting the terms $g_i(y)$ whose correspondence OR_i is less than a base value OR_b the expression (37) is simplified. For $OR_b=0.0001$ we have following function.

$$\begin{aligned} u(y) &= 0.099239-0.05342\,y^2 \\ &-0.039622y^4-8.7495\times10^{-3}y^6 \\ &+1.3474\times10^{-3}y^8+1.2274\times10^{-3}y^{10} \end{aligned} \quad (41)$$

C. *Convergence of the HAM solution*

The convergence region and rate of solution series can be adjusted and controlled by means of the auxiliary parameter ħ, as pointed by Liao [11]. In general, by means of the so-called ħ-curve, it is straightforward to choose an appropriate range for ħ which ensures the convergence of the solution series. To influence of ħ on the convergence of solution, we plot the so-called ħ-curve of $u''(0)$ by 20th-order approximation, as shown in Fig. 2-5. The solutions converge for ħ values which are corresponding to the horizontal line segment in ħ curve. It is easy to discover that(ħ=-0.14)is suitable value which is used for values of $(M \geq 0)$and $(0 \leq T \leq 6)$.

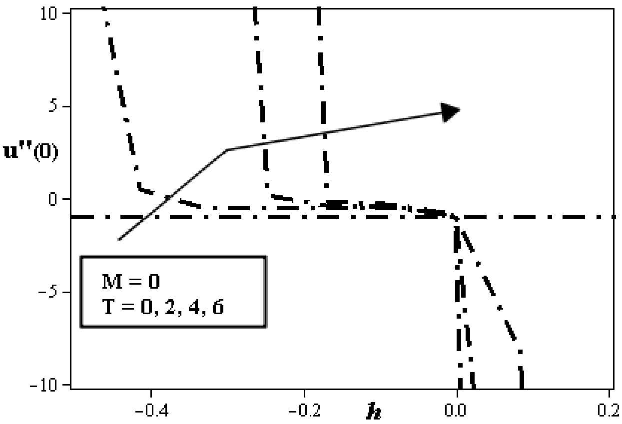

Fig. 2. The $\hbar$ - validity of $u''(0)$ for various value of T when $M = 0$

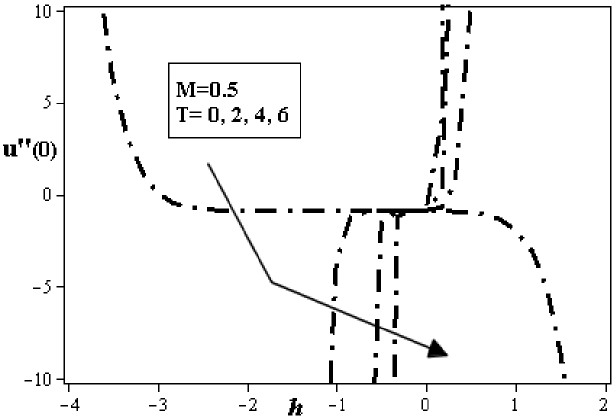

Fig. 3. The $\hbar$ - validity of $u''(0)$ for various value of T when $M = 0.5$

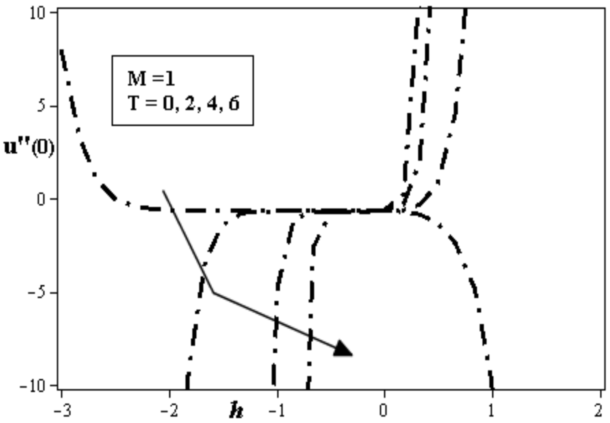

Fig. 4. The $\hbar$ - validity of $u''(0)$ for various value of T when $M = 1$

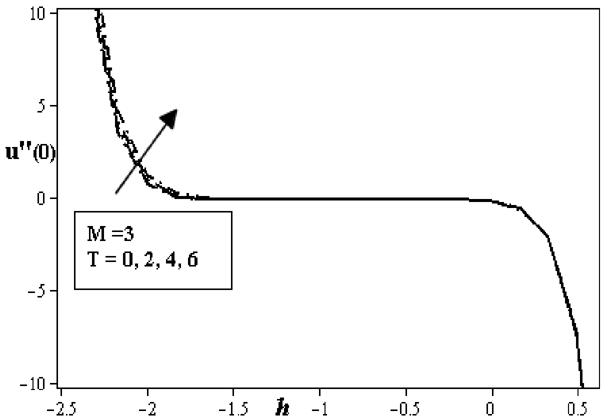

Fig. 5. The $\hbar$ - validity of $u''(0)$ for various value of T when $M = 3$

D. Results and discussion

The present study works on new algorithm through which by adding the number of boundary conditions and auxiliary linear operator order in HAM, the initial guess function will have better convergence to final solution. The main Eq.(16) with boundary conditions (17) is solved by numerical method. Figures 6 – 11 show comparison between the numerical and HAM solutions for $u(y)$ with different values of (T) and (M) . According to Figures 6 – 11, HAM led to appropriate results for nonlinear problems.

Figures 6 to 11 are also prepared in order to see the effects of dimensionless non-Newtonian constant parameter (T) and Hartman number (M) on the velocity profiles. Figures 6 and 7, show the effects of the Hartman number on the velocity profiles. It can be seen that for constant parameters T, the velocity profiles are sharpen with decreasing the M due to magnetic intensity decrease. Figures 8 to 11, show the effects of the T on the velocity profiles. It is found that the velocity distribution is more uniform with increasing T. Besides, the results show that the curves $(T = 6)$ approaches to curves (T=0) by increasing M. Subsequently results indicate that the behavior of third grade fluid flow approaches the Newtonian one with increasing the magnetic field strength.

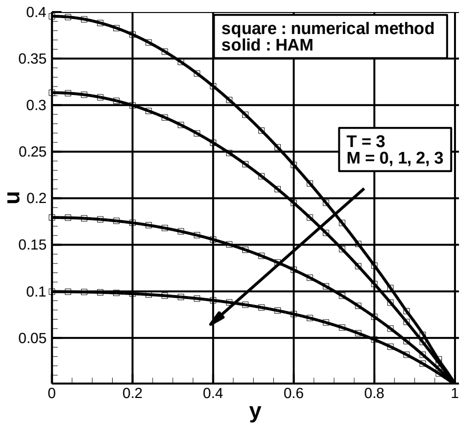

Fig.6. The result of $u(y)$ for various M when $T = 3$

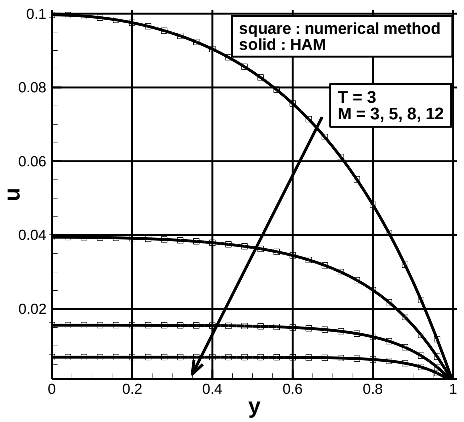

Fig.7. The result of $u(y)$ for various M when $T = 3$

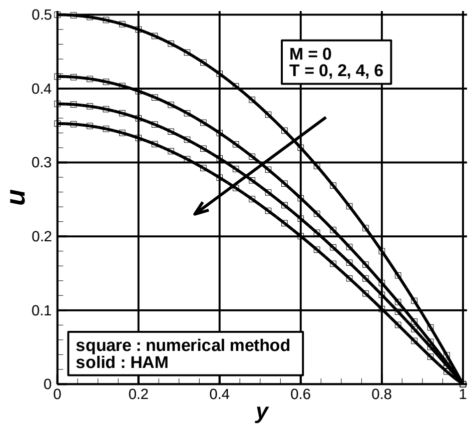

Fig.8. The result of $u(y)$ for various T when $M = 0$

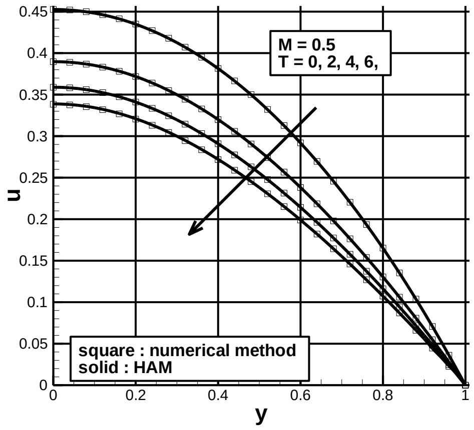

Fig.9. The result of $u(y)$ for various T when $M = 0.5$

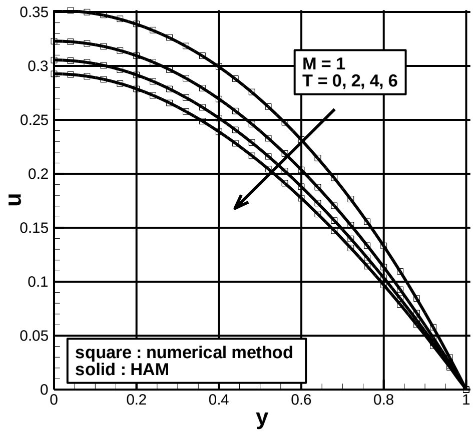

Fig.10. The result of $u(y)$ for various T when $M = 1$

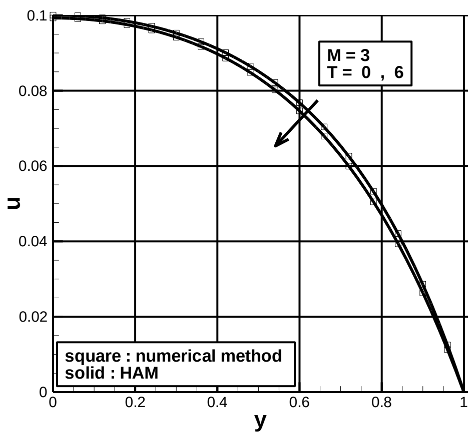

Fig.11. The result of $u(y)$ for various T when $M = 3$

II. Conclusion

In the present study, the MHD fully developed third grade fluid flow in a channel has been considered by using Homotopy Analysis Method. In comparison with other works, we use a new algorithm to guess an initial function through which the convergence series solution can be found faster. It is noteworthy; this technique can be used in similar methods. The obtained analytical solution in comparison with the numerical ones represents a remarkable accuracy. Graphical results are presented to investigate the effects of physical parameters Hartman number (M) and dimensionless non-Newtonian constant parameter (T) on the velocity profiles. The following remarkable results can be concluded:

- It is shown that the more suitable guess function obtained by differentiating main equation and introducing additional boundary condition.
- It is illustrated that by using order of magnitude definition, the series solution is simplified which is useful for analyzing energy equation.
- The results indicate that the behavior of third grade fluid flow approached the Newtonian one with increasing the magnetic field strength.
- It can be seen that for constant parameters T, the velocity profiles are sharpen with decreasing the M.
- It is illustrated that the velocity distribution is more uniform with increasing T.

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