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CONTINUOUS P-FRAMES AND THEIR PERTURBATION IN BANACH SPACES

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***Abstract*— Replacing the sequence of vectors with a net indexed by an ordered set where the set is endowed with a measure space, we obtain a generalization of discrete frames which is called continuous p-frames. The problem of combining the synthesis and analysis operators of these frames is solved in this paper. We also prove that a perturbation of a weakly measurable function G of a cp-frame F is again a cp-frame when there is a small enough gap between F and G.**

***Index Terms*— : Continuous p-frames, Duality mapping, Perturbation**

I. INTRODUCTION

A discrete frame is a countable family of elements in a separable Hilbert space which allows stable not necessarily unique decomposition of arbitrary elements into expansions of the frame elements. This concept was generalized by Ali, Antoine and Gazeau [1], to families indexed by an ordered set endowed with a Radon measure. These frames are known as continuous frames. For more studies about frame theory and continuous frames we refer to [1, 3, 4, 5]. We observe that various generalizations of frames have been proposed recently.

Throughout this paper, (Ω,μ) will be a measure space and μ is a positive, σ-finite measure. X is a Banach space with dual X^*. We choose $1<p<\infty$, and q such that $\frac{1}{p}+\frac{1}{q}=1$. The normed dual X^* of a Banach space X is itself a Banach space and hence has a normed dual of its own, denoted by X^{**}. The mapping $\Lambda_X:X\to X^{**}$, $x\to\Lambda_X x$ defines a unique $\Lambda_X x\in X^{**}$ by the equation, $\langle x,x^*\rangle=\langle x^*,\Lambda_X x\rangle$ for each $x^*\in X^*$ and $\|\Lambda_X x\|=\|x\|$ for each $x\in X$. So $\Lambda_X:X\to X^{**}$ is an isometric isomorphism of X onto a closed subspace of X^{**}. If X is a reflexive Banach space then $\Lambda_X:X\to X^{**}$ is an isometric isomorphism of X onto X^{**}.

A. 2 PRELIMINARIES

Definition 2.1. *A countable family $\{g_i\}_{i=1}^{\infty}\subset X^*$ is a p-frame for X if there exist constants A,B>0 such that*

$$A\|f\|\leq(\sum_{i=1}^{\infty}|g_i(f)|^p)^{\frac{1}{p}}\leq B\|f\|.$$

$\{g_i\}_{i=1}^{\infty}$ is a p-Bessel sequence if at least the upper p-frame condition is satisfied.

Definition 2.2. *Let H be a complex Hilbert space and (Ω,μ) be a measure space. The mapping $F:\Omega\to H$ is called a continuous frame for H with respect to (Ω,μ), if:*

(i) F is weakly measurable, i.e., for each $f\in H$, $\omega\to\langle f,F(\omega)\rangle$ is a measurable function on Ω,

(ii) There exist constants A,B>0 such that

$$A\|f\|^2\leq\int_{\Omega}|\langle f,F(\omega)\rangle^2|d\mu(\omega)\leq B\|f\|^2,\ f\in H. \tag{2.2}$$

Now we recall some theorems and lemmas which we use in this paper.

Lemma 2.3. [8]. *Suppose X and Y are Banach spaces and $T\in B(X,Y)$. Then $R(T)=Y$ if and only if $\|T^*y^*\|\geq c\|y^*\|$ for some constant c>0 and for each $y^*\in Y^*$.*

Theorem 2.4. [9]. *$L^p(\Omega,\mu)$ is isometrically isomorphism to the dual space of $L^q(\Omega,\mu)$ by the mapping* $K^p:L^p(\Omega,\mu)\to L^q(\Omega,\mu)^*$,

$K^p\psi(\phi)=\int_\Omega \psi(\omega)\phi(\omega)d\mu(\omega)$ *for all* $\psi\in L^p(\Omega,\mu)$ *and* $\phi\in L^q(\Omega,\mu)$.

We can define the isometrical isomorphism $K^q=(K^p)^*\Lambda_q:L^q(\Omega,\mu)\to L^p(\Omega,\mu)^*$ *, for which* Λ_q *is the isometrical isomorphism of* $L^q(\Omega,\mu)$ *onto* $L^q(\Omega,\mu)^{**}$.

Lemma 2.5. [7]. *Given a bounded operator* $U:X\to Y$*, the adjoint* $U^*:Y^*\to X^*$ *is surjective if and only if* U *has a bounded inverse on its range* $R(U)$.

B. 3 CP-FRAMES

Definition 3.1. *The mapping* $F:\Omega\to X^*$ *is called a continuous p-frame or a cp-frame for* X *with respect to* (Ω,μ) *if:*

(i) F *is weakly measurable, i.e., for each* $x\in X$, $w\to\langle x,F(\omega)\rangle=F(\omega)(x)$ *is measurable on* Ω.

(ii) There exist positive constants A *and* B *such that*

$$A\|x\|\le\left(\int_\Omega|\langle x,F(\omega)\rangle|^p d\mu(\omega)\right)^{\frac{1}{p}}\le B\|x\| \qquad (3.1)$$

The constants A *and* B *are called the lower and upper cp-frame bounds, respectively.* F *is called a tight cp-frame if* A *and* B *can be chosen such that* $A=B$*, and a Parseval cp-frame if* A *and* B *can be chosen such that* $A=B=1$. F *is called a cp-Bessel mapping for* X *with respect to* (Ω,μ)*, if (i) and the second inequality in (3.1) holds. In this case* B *is called cp-Bessel constant.*

If in the definition of a cp-frame, the measure space $\Omega=N$ and μ be the counting measure, then our cp-frame will be a p-frame and so we expect that some properties of p-frames can be satisfied in cp-frames.

Throughout this paper, we simply say F is a cp-frame for X and F is a cp-Bessel mapping for X, instead of F is a cp-frame for X with respect to (Ω,μ) and F is a cp-Bessel mapping for X with respect to (Ω,μ), respectively.

Our study of a cp-frame is based on analysis of two operators $U_F:X\to L^p(\Omega,\mu)$, defined by

$$U_F x(\omega)=\langle x,F(\omega)\rangle, x\in X,\omega\in\Omega, \qquad (3.2)$$

and $T_F:L^q(\Omega,\mu)\to X^*$ which is weakly defined by

$$T_F\phi(x)=\langle x,T_F\phi\rangle=\int_\Omega\phi(\omega)\langle x,F(\omega)\rangle d\mu(\omega),\phi\in L^q(\Omega,\mu),x\in X \qquad (3.3)$$

It is clear that if F is a cp-Bessel mapping, then U_F is well-defined and bounded operator. U_F is called the analysis and T_F is called the synthesis operator of F.

Lemma 3.2. *Let* F *be a cp-frame for* X*. Then the operator* $U_F:X\to L^p(\Omega,\mu)$ *, given by (3.2), has a closed range and* X *is reflexive.*

Proof. *It is easy to verify that* U_F has a closed range. By the cp-frame condition, X is isomorphic to $R(U_F)$, but $R(U_F)$ is reflexive because it is a closed subspace of the reflexive space $L^p(\Omega,\mu)$ *and therefore* X *is reflexive.*

Theorem 3.3 *Let* $F:\Omega\to X^*$ *be a cp-Bessel mapping for* X *with Bessel bound* B*. Then the operator* $T_F:L^q(\Omega,\mu)\to X^*$ *, weakly defined in (3.3), is well-defined, linear and* $\|T_F\|\le B$.

Lemma 3.4. *Let* $F:\Omega\to X^*$ *be a cp-Bessel mapping for* X*. Then:*

(i) $U_F^*=T_F(K^q)^{-1}$.

(ii) If X *is reflexive, then* $T_F^*=K^pU_F\Lambda_X^{-1}$.

Theorem 3.5 *Let* X *be a reflexive Banach space and* $F:\Omega\to X^*$ *be weakly measurable. If the mapping* $T_F:L^q(\Omega,\mu)\to X^*$ *weakly defined by*

$$\langle x,T_F\phi\rangle=\int_\Omega\phi(\omega)\langle x,F(\omega)\rangle d\mu(\omega),\phi\in L^q(\Omega,\mu),x\in X.$$

is a bounded operator and $\|T_F\|\le B$*, then* F *is a cp-Bessel mapping for* X.

Proof. Since T_F *is well-defined and bounded, for all* $f \in X^*$ *and* $\varphi \in L^q(\Omega,\mu)$, *we have*

$$\langle \varphi, T_F^* f\rangle = \langle T_F\varphi, f\rangle = \int_\Omega \varphi(\omega)\langle \Lambda_X^{-1} f, F(\omega)\rangle d\mu(\omega).$$

For each $f \in X^{**}$, *we define* $\psi_f : \Omega \to C, \omega \to \langle \Lambda_X^{-1} f, F(\omega)\rangle$. *Since* Ψ_f *is measurable and for each* $\varphi \in L^q(\Omega,\mu)$,

$$\left|\int_\Omega \varphi(\omega)\psi_f(\omega)d\mu(\omega)\right| < \infty,$$

$\psi_f \in L^p(\Omega,\mu)$, *by Theorem 2.4, we have*

$$\psi_f(\omega) = (K^p)^{-1}(T_F^* f)(\omega), \omega \in \Omega..$$

Hence for each $x \in X$,

$$(\int_\Omega |\langle x, F(\omega)\rangle|^p d\mu(\omega))^{\frac{1}{p}} = \|(K^p)^{-1}T_F^*\Lambda_X x\| = \|T_F^*\Lambda_X x\|$$
$$\le \|T_F^*\|\|x\| \le B\|x\|.$$

Theorem 3.6. *Let X be a reflexive Banach space and* $F:\Omega\to X^*$ *be a weakly measurable mapping. Then F is a cp-frame for X if and only if* T_F *is a well-defined and bounded operator of* $L^q(\Omega,\mu)$ *onto* X^*. *In this case, the frame bounds are* $\|(T_F^*)^{-1}\|^{-1}$ *and* $\|T_F\|$.

Proof. By Theorem 3.3 and 3.5, the upper cp-frame condition satisfies if and only if T_F *is well-defined and bounded operator of* $L^q(\Omega,\mu)$ *onto* X^*. *Now suppose that F is a cp-feame for X. Then* U_F *has a bounded inverse on its range* $R(U_F)$ *and by Lemma 2.5,* U_F^* *is surjective and therefore* T_F *is a well-defined and bounded operator of* $L^q(\Omega,\mu)$ *onto* X^*. *By Lemma 3.4, for each* $x \in X$,

$$\|U_F x\| = \|(K^p)^{-1}T_F^*\Lambda_X x\| = \|T_F^*\Lambda_X x\| \le \|T_F\|\|x\|.$$

On the other hand since T_F *is bounded and surjective.* T_F^* *is one to one, hence* T_F^* *has a bounded inverse on* $R(T_F^*)$. *So by Lemma 3.4, for each* $x \in X$ *we have*

$$\|x\| = \|\Lambda_X x\| = \|(T_F^*)^{-1}T_F^*\Lambda_X x\| \le \|(T_F^*)^{-1}\|\|U_F x\|.$$

C. 4 CP-FRAME MAPPING AND ITS INVERTIBILITY

In this section, in order to make a cp-frame mapping, we need a mapping from the Banach space $L^p(\Omega,\mu)$ into it's dual space, $L^q(\Omega,\mu)$. For this aim we use the concept of duality mapping.

Definition 4.1. *The mapping* ϕ_X *of X into the set of subsets of* X^*, *defined by*

$$\phi_X x = \{x^* \in X^* : x^*(x) = \|x\|\|x^*\|, \|x^*\| = \|x\|,$$

is called the duality mapping on X.

By the Hahn-Banach theorem, for each $x\in X$, $\phi_X x$ is nonempty and $\phi_X 0=0$. In general the duality mapping is set-valued, but for certain spaces it is single-valued and such spaces are called smooth.

Definition 4.2. *Let* $F:\Omega\to X^*$ *be a cp-frame for X. The bounded mapping* $S_F:X\to X^*$ *defined by* $S_F=T_F(K^q)^{-1}\phi_{L^p(\Omega,\mu)}U_F$ *will be called a cp-frame mapping of F.*

Proposition 4.3. *Suppose that* $F:\Omega\to X^*$ *is a cp-frame for X with frame bounds A and B. Then* S_F *has the following properties:*

(i) $S_F=U_F^*\phi_{L^p(\Omega,\mu)}U_F$.

(ii) $A^2\|x\|^2 \le S_F x(x) \le B^2\|x\|^2, x \in X..$

Definition 4.4. *A mapping [.,.] from X×X into R is said to be a semi-inner product on X if it has these properties:*

(i) $[x,x]\ge 0$ *for all* $x\in X$ *and* $[x,x]=0$ *iff* $x=0$.

(ii) $[\alpha x+\beta y,z]=\alpha[x,z]+\beta[y,z]$ *for all* $\alpha,\beta\in R$ *and for all* $x,y,z\in X$.

(iii) $|[x,y]|^2\le[x,x][y,y]$ *for al l* $x,y\in X$.

The element $x\in X$ is called (Giles) orthogonal to the element $y\in X$ (denoted by $x\perp y$), if $[y,x]=0$. If M is a linear subspace of X, the notation $M^\perp$ is used to show the orthogonal complement of M in Giles sense, i.e. $M^\perp=\{x\in X;\ x\perp y,\ y\in M\}$.

Remark 4.5. *Let $F:\Omega\to X^*$ be a cp-frame for X. Suppose that $Ker(T_F)$ and $(Ker(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega,\mu)$, then clearly the operator $T_F|_{(Ker(T_F))^{\perp}}$ is invertible and $T_F^{\perp}=(T_F|_{(Ker(T_F))^{\perp}})^{-1}$ is a bounded right inverse of T_F.*

Definition 4.6. *Let $F:\Omega\to X^*$ be a cp-frame for X. Suppose that $Ker(T_F)$ and $(Ker(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega,\mu)$, we define the mapping $K:X^*\to X$ by $K=\Lambda_X^{-1}(T_F^{\perp})^*\phi_{L^q(\Omega,\mu)}T_F^{\perp}$.*

Lemma 4.7. *Let $F:\Omega\to X^*$ be a cp-frame for X. Suppose that $Ker(T_F)$ and $(Ker(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega,\mu)$. Then:*

(i) $K(g)(g)\geq\frac{1}{B^2}\|g\|_X^2$, where B denotes an upper cp-frame bound for F.

Moreover, when the operator $T_F^{\perp}T_F$ is adjoint abelian, the following assertions hold:

(ii) S_F is invertible and $S_F^{-1}=K$.

(iii) $S_F^{-1}=U_F^{-1}(K^p)^{-1}\phi_{L^q(\Omega,\mu)}T_F^{\perp}$.

D. 5 DUALS OF CP-BESSEL MAPPINGS

In this section, X is an infinite dimensional, reflexive Banach space.

Definition 5.1. [6]. *A sequence $\{e_i\}_{i=1}^{\infty}$ in X is called a Schauder basis of X, if for each $x\in X$ there is a unique sequence of scalars $(a_i)_{i=1}^{\infty}$, called the coordinates of x, such that $x=\sum_{i=1}^{\infty}a_ie_i$.*

Let $\{e_i\}_{i=1}^{\infty}$ be a Schauder basis of a Banach space X.

For $j\in N$ and $x=\sum_{i=1}^{\infty}a_ie_i$, denote $f_j(x)=a_j$. Using Theorem 6.5 in [6], $f_j\in X^*$. The functionals $\{f_i\}_{i=1}^{\infty}$ are called the associated biorthogonal functionals (coordinate functionals) to $\{e_i\}_{i=1}^{\infty}$ and for each $x\in X$, we have $x=\sum_{i=1}^{\infty}f_i(x)e_i$.

We will denote the biorthogonal functionals $\{f_i\}$ by $\{e_i^*\}$, and say that $\{e_i,e_i^*\}$ is a Schauder basis of X.

Theorem 5.2 *Let $F:\Omega\to X^*$ be a cp-Bessel mapping for X and $G:\Omega\to X^{**}$ be a cq-Bessel mapping for X^*. Then the following assertions are equivalent:*

(i) For each $x\in X$, $x=\Lambda_X^{-1}T_G(K^p)^{-1}T_F^*\Lambda_X x$.

(ii) For each $g\in X^*$, $g=T_F(K^q)^{-1}T_G^*(\Lambda_X^*)^{-1}g$.

(iii) For each $x\in X$ and $g\in X^*$ $\langle x,g\rangle=\int_{\Omega}\langle x,F(\omega)\rangle\langle g,G(\omega)\rangle d\mu(\omega)$.

(iv) For each Schauder basis $\{e_i,e_i^*\}$ of X,

Error!

Definition 5.3. *Let $F:\Omega\to X^*$ be a cp-Bessel mapping for X and $G:\Omega\to X^{**}$ be a cq-Bessel mapping for X^*. We say that (F,G) is a c-dual pair, if one of the assertions of Theorem 5.25, satisfies.*

In this case F is called a cp-dual of G and by Theorem 5.2, we can say that G is a cq-dual of F.

Definition 5.4. *Let $F:\Omega\to X^*$ be a cp-frame for X. We say that F is independent, provident that for each measurable function $\phi:\Omega\to C$ and $x\in X$,*

$$\int_{\Omega}\langle x,F(\omega)\rangle\phi(\omega)d\mu(\omega)=0,$$

implies that $\phi=0$.

Theorem 5.5 *Let $F:\Omega\to X^*$ be a cp-frame for X and $\mu(E)\geq k>0$, for each measurable set E, except $E=\varnothing$. Then, we have the following assertions:*

(i) If F is an independent cp-frame for X, then there exists a unique cq-frame, $G:\Omega\to X^{**}$ for X^*, such that (F,G) is a c-dual pair.

(ii) If $Ker(T_F)$ and $(Ker(T_F))^{\perp}$ are topologically complementary in $L^q(\Omega,\mu)$, then there exists a cq-

frame $G:\Omega\to X^{**}$ for X^*, such that (F,G) is a c-dual pair.

E. 6 PERTURBATION OF CP-FRAMES

Perturbation of discrete frames has been discussed in [2]. The proof of the following theorem is based on the following lemma, which was proved in [2].

Lemma 6.1. *Let U be a linear operator on a Banach space X and assume that there exist $\lambda_1,\lambda_2\in[0,1)$ such that for each $x\in X$,*

$$|x-Ux|\le\lambda_1|x|+\lambda_2|Ux|.$$

Then U is bounded and invertible. Moreover for each $x\in X$,

$$\frac{1-\lambda_1}{1+\lambda_2}|x|\le|Ux|\le\frac{1+\lambda_1}{1-\lambda_2}|x|,$$

and

$$\frac{1-\lambda_2}{1+\lambda_1}|x|\le|U^{-1}x|\le\frac{1+\lambda_2}{1-\lambda_1}|x|.$$

Theorem 6.2 *Let F be an independent cp-frame for X and $\mu(E)\ge k>0$, for each measurable set E, except $E=\varnothing$. Suppose that $G:\Omega\to X^*$ is weakly measurable and assume that there exist constants $\lambda_1,\lambda_2,\gamma\ge 0$ such that $\max(\lambda_1+\frac{\gamma}{A},\lambda_2)<1$. Let for all $\phi\in L^q(\Omega,\mu)$ and x in the unit sphere of X,*

$$\left|\int_\Omega \phi(\omega)\langle x,F(\omega)-G(\omega)\rangle d\mu(\omega)\right| \le \lambda_1\left|\int_\Omega \phi(\omega)\langle x,F(\omega)\rangle d\mu(\omega)\right| + \lambda_2\left|\int_\Omega \phi(\omega)\langle x,G(\omega)\rangle d\mu(\omega)\right| + \gamma\|\phi\|$$

Then $G:\Omega\to X^*$ is a cp-frame for X with bounds

$$A\left[\frac{1-(\lambda_1+\frac{\gamma}{A})}{1+\lambda_2}\right] \quad \text{and} \quad B\left[\frac{1+\lambda_1+\frac{\gamma}{B}}{1-\lambda_2}\right],$$

where A and B are the frame bounds of F.

REFERENCES

S. T. Ali, J. P. Antoine and J. P. Gazeau, Continuous frames in Hilbert spaces, Ann. Phys. 222 (1993), 1-37.

[2] P. G. Casazza and O. Christensen, Perturbation of operators and applications to frame theory, J. Fourier Anal. Appl. 3 (1997), 543-557.

[3] O. Christensen, An Introduction to Frames and Riesz Bases, Birkhauser, Boston, 2003.

[4] H. Deguang and J. P. Gabardo, Frames associated with measurable spaces, Adv. Comp. Math. 18 (2003), 127-147.

[5] Y. N. Dehghan, A. Najati and A. Rahimi, Continuous frame in Hilbert space, Methods Funct. Anal. Topology. 12 (2006), 170-182.

[6] M. Fabian, P. Habala, P. Hajek, J. Pelant and V. Zizler, Functional Analysis and Infinite Dimensional Geometry, Springer, New York, 1972.

[7] H. Heuser, Functional Analysis. John Wiley, New York, 1982.

[8] W. Rudin, Functional Analysis, Mc Graw-Hill, New York, 1973.

[9] W. Rudin, Real and Complex Analysis, Mc Graw-Hill, New York, 1974.

[1] [10] D. T. Stoeva, Generalization of the frame operator and the canonical dual frame to Banach spaces, Asian-European Journal of Mathematics. 1 (2008), 631-643..