MATHEMATICAL MODEL OF SPATIALLY LOADED BARS WITH ACCOUNT OF TORSION FUNCTION AND TRANSVERSE SHEARS

Sh.A. Anarova[1], F.M. Nuraliev[2], G.K.Dadenova[3]
[1]Candidate of Technical Sciences,.Senior Research Worker, Centre of Development of Software and Apparatus-Program Complexes
[2]Candidate of Physical and Mathematical Sciences, Senior Lecturer
[3]Assistant Department of foundations of Informatics
Tashkent University of Information Technologies
omon_shoira@mail.ru

Abstract: ***A derivation of mathematical model of spatially loaded bars with account of torsion function and transverse shears is considered in the paper. On the basis of the problem, with the use of Hamilton-Ostrogradsky's principle the variations of kinetic, potential energy and the work of external forces have been determined; and a mathematical model of spatially loaded bars with account of torsion function and transverse shears has been derived. On the basis of the model, the determinant equations of spatially loaded bars have been derived with natural initial and boundary conditions.***
A sequence of solution of the equations obtained by Hamilton-Ostrogradsky's variation principle is given in the conclusion of the paper.

Keywords: ***Mathematical model, bar, torsion, transverse shears, spatially loaded, Hamilton-Ostrogradsky's variation principle.***

I. INTRODUCTION

As known, the calculation of thin-wall bars is more complex than that of solid ones. Thin-wall structures are the best to meet the requirements of economy providing at that respective strength and rigidities. This explains their wide use in different spheres of engineering – in machine-building, construction, aviation, etc.

1. Statement of the problem. Applied theory of bar vibration is built on the basis of a number of static and kinematic hypotheses in respect to the law of distribution of displacements, strains and stress in bar cross sections. The problems of bars vibrations discussed below are solved on the basis of the following hypotheses:

1. Under tension, compression and pure bending of the bars of arbitrary profile and under torsion of circular bars all sections, initially plane, remain plane during the process of bar strain (according to the hypothesis of plane sections).

2. Tangent stresses in all cross sections are distributed by one and the same law.

3. Stress components are non-zero. However, due to their small value compared with stress components they are negligible.

4. Efforts applied to the side surface of the beam and volume forces may be substituted by the distributed efforts and moments applied along the axis of the bar.

When forming the applied theory, the transfer from the study of vibration of three-dimensional body to one-dimensional one is of crucial importance. Complete solution of this problem may be obtained from discrete-continual method, developed by V.Z.Vlasov, G.Yu.Djanelidze and V.K.Kabulov.

On the basis of assumptions 1□4, with more specified theory of Vlasov-Djanelidze-Kabulov the expressions for displacements points of the bar under joint longitudinal, transverse and torsional vibrations are given in the form [1-6].

$$\left.\begin{aligned} &u_1(x,y,z)=u-z\alpha_1-y\alpha_2+\varphi(x,y,z)\vartheta+a_1(x,y,z)\beta_1+a_2(x,y,z)\beta_2,\\ &u_2(x,y,z)=v+z\theta,\qquad u_3(x,y,z)=w-y\theta,\end{aligned}\right\} \quad (1.1)$$

where θ – is an angle of twist, u,v,w –displacements of the middle line of the bar, α_1,α_2– section turnover angles under pure bending, β_1,β_2– an angle of transverse shear, u_1,u_2,u_3 – components of displacement vector, x,y,z– spatial variables.

There are twelve sought for functions: $u,v,w,\alpha_1,\alpha_2,\theta,\vartheta,\beta_1,\beta_2,\varphi,a_1,a_2$ and no limitations on external load; functions $u,v,w,\vartheta,\theta,\alpha_1,\alpha_2,\beta_1,\beta_2$ are the functions on spatial variable x and time t.

The formation of the theory of bars may be generalized in two directions. First, coordinate functions may be considered as unknown ones and to define them the corresponding differential equations should be derived from Hamilton-Ostrogradsky principle. This theory is conventionally called a «one-dimensional» one.

The other way to generalize the vibrations of the bar leads to the solution of the problem of mathematical theory of elasticity with strict consideration of boundary conditions.

In general, Hamilton-Ostrogradsky's variation principle [6,12,13] is written:

$$\delta\int_{t_1}^{t_2}(K-\Pi+A)dt=0, \qquad (1.2)$$

where К – is a kinetic and Π – a potential energy; A – a work of external volume and surface forces.

2. Determination of the variation of kinetic energy. In calculating the variations of kinetic energy we use the relation [6-13]:

$$\int_t \delta K dt=\int_t\int_V\left(\rho\frac{\partial u_1}{\partial t}\delta\frac{\partial u_1}{\partial t}+\rho\frac{\partial u_2}{\partial t}\delta\frac{\partial u_2}{\partial t}+\rho\frac{\partial u_3}{\partial t}\delta\frac{\partial u_3}{\partial t}\right)dvdt, \qquad (2.1)$$

where ρ – is a specific density of mass of the body material (assumed to be constant).

Integrating by parts, substituting expressions u_1,u_2,u_3 from (1.1) into variations of kinetic energy (2.1) and conducting corresponding analytical computation (integration, differentiation and reduction of similar terms), with account of introduced indications we get the variations of kinetic energy in the form:

$$\int_t Kdt=\int_x\Bigg\{\Bigg(\rho F\frac{\partial u}{\partial t}-\rho S_y\frac{\partial\alpha_1}{\partial t}-\rho S_z\frac{\partial\alpha_2}{\partial t}+\int_y\int_z\left(\rho\varphi\frac{\partial\vartheta}{\partial t}+\rho a_1\frac{\partial\beta_1}{\partial t}+\rho a_2\frac{\partial\beta_2}{\partial t}+\rho\vartheta\frac{\partial\varphi}{\partial t}+\right.$$

$$\left.+\rho\beta_1\frac{\partial a_1}{\partial t}+\rho\beta_2\frac{\partial a_2}{\partial t}\right)dzdy\Bigg)\delta u+\left(\rho F\frac{\partial v}{\partial t}+\rho S_y\frac{\partial\theta}{\partial t}\right)\delta v+\left(\rho F\frac{\partial w}{\partial t}-\rho S_z\frac{\partial\theta}{\partial t}\right)\delta w+$$

$$-\Bigg(\rho S_y\frac{\partial u}{\partial t}-\rho J_y\frac{\partial\alpha_1}{\partial t}-\rho J_{zy}\frac{\partial\alpha_2}{\partial t}+\int_y\int_z\left(\rho z\varphi\frac{\partial\vartheta}{\partial t}+\rho za_1\frac{\partial\beta_1}{\partial t}+\rho za_2\frac{\partial\beta_2}{\partial t}+\rho z\vartheta\frac{\partial\varphi}{\partial t}+\rho z\beta_1\frac{\partial a_1}{\partial t}+\right.$$

$$\left.+\rho z\beta_2\frac{\partial a_2}{\partial t}\right)dzdy\Bigg)\delta\alpha_1-\Bigg(\rho S_z\frac{\partial u}{\partial t}-\rho J_{yz}\frac{\partial\alpha_1}{\partial t}-\rho J_z\frac{\partial\alpha_2}{\partial t}+$$

$$+\int_y\int_z\left(\rho y\varphi\frac{\partial\vartheta}{\partial t}+\rho ya_1\frac{\partial\beta_1}{\partial t}+\rho ya_2\frac{\partial\beta_2}{\partial t}+\rho y\vartheta\frac{\partial\varphi}{\partial t}+\rho y\beta_1\frac{\partial a_1}{\partial t}+\rho y\beta_2\frac{\partial a_2}{\partial t}\right)dzdy\Bigg)\delta\alpha_2+$$

$$+\left(\rho S_z\frac{\partial w}{\partial t}-\rho S_y\frac{\partial v}{\partial t}+\rho J_p\frac{\partial\theta}{\partial t}\right)\delta\theta+\int_y\int_z\left(\rho\varphi\frac{\partial u}{\partial t}-\rho\varphi z\frac{\partial\alpha_1}{\partial t}-\rho\varphi y\frac{\partial\alpha_2}{\partial t}+\rho\varphi^2\frac{\partial\vartheta}{\partial t}+\rho\varphi a_1\frac{\partial\beta_1}{\partial t}+\right.$$

$$\left.+\rho\varphi a_2\frac{\partial\beta_2}{\partial t}+\rho\varphi\vartheta\frac{\partial\varphi}{\partial t}+\rho\varphi\beta_1\frac{\partial\alpha_1}{\partial t}+\rho\varphi\beta_2\frac{\partial\alpha_2}{\partial t}\right)dzdy\delta\vartheta+$$

$$+\int_y\int_z\left(\rho a_1\frac{\partial u}{\partial t}-\rho a_1 z\frac{\partial\alpha_1}{\partial t}-\rho a_1 y\frac{\partial\alpha_2}{\partial t}+\rho\varphi a_1\frac{\partial\vartheta}{\partial t}+\rho a_1{}^2\frac{\partial\beta_1}{\partial t}+\rho a_1a_2\frac{\partial\beta_2}{\partial t}+\rho a_1\vartheta\frac{\partial\varphi}{\partial t}+\right.$$

$$\left.+\rho a_1\beta_1\frac{\partial\alpha_1}{\partial t}+\rho a_1\beta_2\frac{\partial\alpha_2}{\partial t}\right)dzdy\delta\beta_1+$$

$$+\int_y\int_z\left(\rho a_2\frac{\partial u}{\partial t}-\rho a_2 z\frac{\partial\alpha_1}{\partial t}-\rho a_2 y\frac{\partial\alpha_2}{\partial t}+\rho\varphi a_2\frac{\partial\vartheta}{\partial t}+\rho a_1a_2\frac{\partial\beta_1}{\partial t}+\rho a_2{}^2\frac{\partial\beta_2}{\partial t}+\rho\vartheta a_2\frac{\partial\varphi}{\partial t}+\right.$$

$$\left.+\rho a_2\beta_1\frac{\partial\alpha_1}{\partial t}+\rho a_2\beta_2\frac{\partial\alpha_2}{\partial t}\right)dzdy\delta\beta_2+$$

$$+\Bigg(\rho F\frac{\partial u}{\partial t}-\rho S_y\frac{\partial\alpha_1}{\partial t}-\rho S_z\frac{\partial\alpha_2}{\partial t}+\int_y\int_z\left(\rho\varphi\frac{\partial\vartheta}{\partial t}+\rho a_1\frac{\partial\beta_1}{\partial t}+\rho a_2\frac{\partial\beta_2}{\partial t}+\rho\vartheta\frac{\partial\varphi}{\partial t}+\right.$$

$$\left.+\rho\beta_1\frac{\partial a_1}{\partial t}+\rho\beta_2\frac{\partial a_2}{\partial t}\right)dzdy(\vartheta\delta\varphi+\beta_1\delta a_1+\beta_2\delta a_2)\Bigg\}dx\Bigg|_t-$$

$$-\int_t\int_x\left\{\left(\rho F\frac{\partial^2 u}{\partial t^2}-S_y\rho\frac{\partial^2\alpha_1}{\partial t^2}-S_z\rho\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho\varphi\frac{\partial^2\vartheta}{\partial t^2}+\rho a_1\frac{\partial^2\beta_1}{\partial t^2}+\rho a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.\right.$$

$$\left.\left.+\rho\vartheta\frac{\partial^2\varphi}{\partial t^2}+\rho\beta_1\frac{\partial^2 a_1}{\partial t^2}+\rho\beta_2\frac{\partial^2 a_2}{\partial t^2}+2\rho\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta u+$$

$$+\left(\rho F\frac{\partial^2 v}{\partial t^2}+\rho S_y\frac{\partial^2\theta}{\partial t^2}\right)\delta v+\left(\rho F\frac{\partial^2 w}{\partial t^2}-\rho S_z\frac{\partial^2\theta}{\partial t^2}\right)\delta w+\left(\rho S_y\frac{\partial^2 v}{\partial t^2}-\rho S_z\frac{\partial^2 w}{\partial t^2}+\rho J_\rho\frac{\partial^2\theta}{\partial t^2}\right)\delta\theta-$$

$$-\left(\rho S_y\frac{\partial^2 u}{\partial t^2}-\rho J_y\frac{\partial^2\alpha_1}{\partial t^2}-\rho J_{yz}\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho z\varphi\frac{\partial^2\vartheta}{\partial t^2}+\rho z a_1\frac{\partial^2\beta_1}{\partial t^2}+\rho z a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.+\vartheta\rho z\frac{\partial^2\varphi}{\partial t^2}+\beta_1\rho z\frac{\partial^2 a_1}{\partial t^2}+\beta_2\rho z\frac{\partial^2 a_2}{\partial t^2}+2\rho z\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho z\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho z\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta\alpha_1-$$

$$-\left(\rho S_z\frac{\partial^2 u}{\partial t^2}-\rho J_{yz}\frac{\partial^2\alpha_1}{\partial t^2}-\rho J_z\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho y\varphi\frac{\partial^2\vartheta}{\partial t^2}+\rho y a_1\frac{\partial^2\beta_1}{\partial t^2}+\rho y a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.+\vartheta\rho y\frac{\partial^2\varphi}{\partial t^2}+\beta_1\rho y\frac{\partial^2 a_1}{\partial t^2}+\beta_2\rho y\frac{\partial^2 a_2}{\partial t^2}+2\rho y\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho y\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho y\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta\alpha_2+$$

$$+\left(\rho S_\varphi\frac{\partial^2 u}{\partial t^2}-\rho J_{\varphi z}\frac{\partial^2\alpha_1}{\partial t^2}-\rho J_{\varphi y}\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho\varphi^2\frac{\partial^2\vartheta}{\partial t^2}+\rho\varphi a_1\frac{\partial^2\beta_1}{\partial t^2}+\rho\varphi a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.+\vartheta\rho\varphi\frac{\partial^2\varphi}{\partial t^2}+\beta_1\rho\varphi\frac{\partial^2 a_1}{\partial t^2}+\beta_2\rho\varphi\frac{\partial^2 a_2}{\partial t^2}+2\rho\varphi\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho\varphi\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho\varphi\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta\vartheta+$$

$$+\left(\rho S_{a_1}\frac{\partial^2 u}{\partial t^2}-\rho J_{za_1}\frac{\partial^2\alpha_1}{\partial t^2}-\rho J_{ya_1}\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho\varphi a_1\frac{\partial^2\vartheta}{\partial t^2}+\rho a_1^2\frac{\partial^2\beta_1}{\partial t^2}+\rho a_1 a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.+\vartheta\rho a_1\frac{\partial^2\varphi}{\partial t^2}+\beta_1\rho a_1\frac{\partial^2 a_1}{\partial t^2}+\beta_2\rho a_1\frac{\partial^2 a_2}{\partial t^2}+2\rho a_1\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho a_1\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho a_1\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta\beta_1+$$

$$+\left(\rho S_{a_2}\frac{\partial^2 u}{\partial t^2}-\rho J_{a_2 z}\frac{\partial^2\alpha_1}{\partial t^2}-\rho J_{a_2 y}\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho a_2\varphi\frac{\partial^2\vartheta}{\partial t^2}+\rho a_2\varphi\frac{\partial^2\beta_1}{\partial t^2}+\rho a_2^2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.+\vartheta\rho a_2\frac{\partial^2\varphi}{\partial t^2}+\beta_1\rho a_2\frac{\partial^2 a_1}{\partial t^2}+\beta_2\rho a_2\frac{\partial^2 a_2}{\partial t^2}+2\rho a_2\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}+2\rho a_2\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}+2\rho a_2\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy\right)\delta\beta_2+$$

$$+\left[\rho F\frac{\partial^2 u}{\partial t^2}-S_y\rho\frac{\partial^2\alpha_1}{\partial t^2}-S_z\rho\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\left(\rho\varphi\frac{\partial^2\vartheta}{\partial t^2}+\rho a_1\frac{\partial^2\beta_1}{\partial t^2}+\rho a_2\frac{\partial^2\beta_2}{\partial t^2}+\right.\right.$$

$$\left.\left.\left.+\rho\vartheta\frac{\partial^2\varphi}{\partial t^2}+\rho\beta_1\frac{\partial^2 a_1}{\partial t^2}+\rho\beta_2\frac{\partial^2 a_2}{\partial t^2}\right)dzdy\right]\times\left(\vartheta\delta\varphi+\beta_1\delta a_1+\beta_2\delta a_2\right)\right\}dxdt. \tag{2.3}$$

Here the following indications have been introduced:

$$\rho F=\int_y\int_z\rho dzdy;\ \rho S_y=\int_y\int_z\rho z dzdy;\ \rho S_z=\int_y\int_z\rho y dzdy;\ \rho J_y=\int_y\int_z\rho z^2 dzdy;$$

$$\rho J_{zy} = \int_y \int_z \rho zydzdy;\ \rho J_z = \int_y \int_z \rho y^2 dzdy;\ \rho J_\rho = \int_y \int_z \rho(z^2 + y^2)dzdy.$$

3. Determination of the variation of potential energy. For the variation of potential energy we have [6,12,13]:

$$\int_v \delta\Pi dv = \int_v (\sigma_{11}\delta\varepsilon_{11} + \sigma_{12}\delta\varepsilon_{12} + \sigma_{13}\delta\varepsilon_{13})dv. \tag{3.1}$$

Here $\sigma_{11}, \sigma_{12}, \sigma_{13}$ - are stress tensors, $\varepsilon_{11}, \varepsilon_{12}, \varepsilon_{13}$ - strain tensors.
On the basis of Cauchy relation we get [6-13]:

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x} = \frac{\partial}{\partial x}(u - z\alpha_1 - y\alpha_2 + \varphi\vartheta + a_1\beta_1 + a_2\beta_2) =$$
$$= \frac{\partial u}{\partial x} - z\frac{\partial \alpha_1}{\partial x} - y\frac{\partial \alpha_2}{\partial x} + \vartheta\frac{\partial \varphi}{\partial x} + \varphi\frac{\partial \vartheta}{\partial x} + \beta_1\frac{\partial a_1}{\partial x} + a_1\frac{\partial \beta_1}{\partial x} + \beta_2\frac{\partial a_2}{\partial x} + a_2\frac{\partial \beta_2}{\partial x};$$
$$\varepsilon_{12} = \frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x} = \frac{\partial}{\partial y}(u - z\alpha_1 - y\alpha_2 + \varphi\vartheta + a_1\beta_1 + a_2\beta_2) +$$
$$+ \frac{\partial}{\partial x}(v + z\theta) = -\alpha_2 + \vartheta\frac{\partial \varphi}{\partial y} + \beta_1\frac{\partial a_1}{\partial y} + \beta_2\frac{\partial a_2}{\partial y} + \frac{\partial v}{\partial x} + z\frac{\partial \theta}{\partial x};$$
$$\varepsilon_{13} = \frac{\partial u_1}{\partial z} + \frac{\partial u_3}{\partial x} = \frac{\partial}{\partial Z}(u - z\alpha_1 - y\alpha_2 + \varphi\vartheta + a_1\beta_1 + a_2\beta_2) +$$
$$+ \frac{\partial}{\partial x}(v - y\theta) = -\alpha_1 + \vartheta\frac{\partial \varphi}{\partial z} + \beta_1\frac{\partial a_1}{\partial z} + \beta_2\frac{\partial a_2}{\partial z} + \frac{\partial w}{\partial x} - y\frac{\partial \theta}{\partial x}. \tag{3.2}$$

Using the variation of potential energy (3.1) and conducting corresponding analytical computation (integration, differentiation and reduction of similar terms), with account of introduced indications we get the variations of potential energy in the form:

$$\int_t \delta\Pi dt = \int_t \Big\{ N_x\delta u + Q_{12}\delta v + Q_{13}\delta w - M_y\delta\alpha_1 - M_z\delta\alpha_2 + M_x\delta\theta +$$
$$+ \int_y \int_z [\vartheta\sigma_{11}\delta\varphi + \beta_1\sigma_{11}\delta a_1 + \beta_2\sigma_{11}\delta a_2 + \varphi\sigma_{11}\delta\vartheta + \beta_1\sigma_{11}\delta a_1 + \beta_2\sigma_{11}\delta a_2\ ]dzdy\Big]_x +$$
$$+ \int_x \int_z [\vartheta\sigma_{12}\delta\varphi + \beta_1\sigma_{12}\delta a_1 + \beta_2\sigma_{12}\delta a_2]dzdx\Big|_y + \int_x \int_y [\vartheta\sigma_{13}\delta\varphi + \beta_1\sigma_{13}\delta a_1 + \beta_2\sigma_{13}\delta a_2]dydx\Big|_z -$$
$$- \int_x \Big[\frac{\partial N_x}{\partial x}\delta u + \frac{\partial Q_{12}}{\partial x}\delta v + \frac{\partial Q_{13}}{\partial x}\delta w - \left(\frac{\partial M_y}{\partial x} - Q_{13}\right)\delta\alpha_1 - \left(\frac{\partial M_z}{\partial x} - Q_{12}\right)\delta\alpha_2 + \frac{\partial M_x}{\partial x}\delta\theta +$$
$$+ \int_y \int_z \left(\frac{\partial}{\partial x}(\varphi\sigma_{11}) - \frac{\partial \varphi}{\partial x}\sigma_{11} - \frac{\partial \varphi}{\partial y}\sigma_{12} - \frac{\partial \varphi}{\partial z}\sigma_{13}\right)\delta\vartheta + \int_y \int_z \left(\frac{\partial}{\partial x}(a_1\sigma_{11}) - \frac{\partial a_1}{\partial x}\sigma_{11} - \frac{\partial a_1}{\partial y}\sigma_{12} - \frac{\partial a_1}{\partial z}\sigma_{13}\right)\delta\beta_1 +$$
$$+ \int_y \int_z \left(\frac{\partial}{\partial x}(a_2\sigma_{11}) - \frac{\partial a_2}{\partial x}\sigma_{11} - \frac{\partial a_2}{\partial y}\sigma_{12} - \frac{\partial a_2}{\partial z}\sigma_{13}\right)\delta\beta_2 + \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z}\right)(\vartheta\delta\varphi + \beta_1\delta a_1 + \beta_2\delta a_2)\Big]dx\Big\}dt. \tag{3.3}$$

Here the following indications are introduced:

$$N_x = \int_y \int_z \sigma_{11}dzdy; \quad Q_{12} = \int_y \int_z \sigma_{12}dzdy; \quad Q_{13} = \int_y \int_z \sigma_{13}dzdy;$$
$$M_y = \int_y \int_z z\sigma_{11}dzdy; \quad M_z = \int_y \int_z y\sigma_{11}dzdy; \quad M_x = \int_y \int_z (\sigma_{12}z - \sigma_{13}y)dzdy.$$

4. Determination of the variation of work of external forces. Consider the variation of work of external forces [6,12,13]:

$$\delta\int_t Adt = \int_v (P_1\delta u_1 + P_2\delta u_2 + P_3\delta u_3)dv + \int_s (q_1\delta u_1 + q_2\delta u_2 + q_3\delta u_3)ds + \int_{s_1}(\varphi_1\delta u_1 + \varphi_2\delta u_2 +$$
$$+\varphi_3\delta u_3)ds_1\big|_x + \int_{s_2}(f_1\delta u_1 + f_2\delta u_2 + f_3\delta u_3)ds_2\big|_y + \int_{s_3}(g_1\delta u_1 + g_2\delta u_2 + g_3\delta u_3)ds_3\big|_z . \qquad (4.1)$$

Using the variation of work of external forces (4.1) and conducting corresponding analytical computation (integration, differentiation and reduction of similar terms), with account of introduced indications we get the variation of work of external forces in the following form:

$$\int_t \delta Adt = \int_t \Bigg\{\int_x \Big[\big(N_x(P_1) + N_x(q_1)\big)\delta u - \big(M_y(P_1) + M_y(q_1)\big)\delta\alpha_1 - \big(M_z(P_1) + M_z(q_1)\big)\delta\alpha_2 +$$
$$+\big(Q_{12}(P_2) + Q_{12}(q_2)\big)\delta v + \big(Q_{13}(P_3) + Q_{13}(q_3)\big)\delta w + \big(M_x(P_2,P_3) + M_x(q_2,q_3)\big)\delta\theta +$$
$$+\left(\int_y\int_z \varphi P_1 dzdy + \int_l \varphi q_1 dl\right)\delta\vartheta + \left(\int_y\int_z a_1P_1dzdy + \int_l a_1q_1dl\right)\delta\beta_1 +$$
$$+\left(\int_y\int_z a_2P_1dzdy + \int_l a_2q_1dl\right)\delta\beta_2 + \left(\int_y\int_z \vartheta P_1dzdy + \int_l \vartheta q_1dl\right)\big(\delta\varphi + \delta a_1 + \delta a_2\big)\Big]dx +$$
$$+\Big[N_x(\varphi_1)\delta u + Q_{12}(\varphi_2)\delta v + Q_{13}(\varphi_3)\delta w - M_y(\varphi_1)\delta\alpha_1 - M_z(\varphi_1)\delta\alpha_2 + M_x(\varphi_2,\varphi_3)\delta\theta +$$
$$+\int_y\int_z\big[\varphi\cdot\varphi_1\delta\vartheta + a_1\varphi_1\delta\beta_1 + a_2\varphi_1\delta\beta_2 + \vartheta\varphi_1\delta\varphi + \beta_1\varphi_1\delta a_1 + \beta_2\varphi_1\delta a_2\big]dzdy\Big]_x +$$
$$+\int_x\int_z\big[\vartheta f_1\delta\varphi + \beta_1 f_1\delta a_1 + \beta_2 f_1\delta a_2\big]dzdx\big|_y + \int_x\int_y\big[\vartheta g_1\delta\varphi + \beta_1 g_1\delta a_1 + \beta_2 g_1\delta a_2\big]dydx\big|_z\Bigg\}dt. \qquad (4.2)$$

5. Derivation of determinant equations of spatially loaded bars. Variations of kinetic (2.3) and potential (3.3) energies and work of external forces (4.2) are substituted into Hamilton-Ostrogradsky's variation principle (1.2).

In variation equation the functions under the signs of variation are unknown. So, variations of these functions are non-zero. Thus, their coefficients should be equal to zero. So, from variation equation we obtain the system of differential equations with natural initial and boundary conditions.

The system of equations of the bar vibrations is:

$$-\rho F\frac{\partial^2 u}{\partial t^2} + S_y\rho\frac{\partial^2\alpha_1}{\partial t^2} + S_z\rho\frac{\partial^2\alpha_2}{\partial t^2} - \int_y\int_z\left(S_\varphi\rho\frac{\partial^2\vartheta}{\partial t^2} + \rho S_{a_1}\frac{\partial^2\beta_1}{\partial t^2} + \rho S_{a_2}\frac{\partial^2\beta_2}{\partial t^2} + \rho\vartheta\frac{\partial^2\varphi}{\partial t^2} +\right.$$
$$\left.+\rho\beta_1\frac{\partial^2 a_1}{\partial t^2} + \rho\beta_2\frac{\partial^2 a_2}{\partial t^2} + 2\rho\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t} + 2\rho\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t} + 2\rho\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy + \frac{\partial N_x}{\partial x} + N_x(P_1) + N_x(q_1) = 0;$$

$$-\rho F\frac{\partial^2 v}{\partial t^2} + \rho S_y\frac{\partial^2\theta}{\partial t^2} + \frac{\partial Q_{12}}{\partial x} + Q_{12}(P_2) + Q_{12}(q_2) = 0;$$

$$-\rho F\frac{\partial^2 w}{\partial t^2} - \rho S_z\frac{\partial^2\theta}{\partial t^2} + \frac{\partial Q_{13}}{\partial x} + Q_{13}(P_3) + Q_{13}(q_3) = 0;$$

$$\rho S_y\frac{\partial^2 u}{\partial t^2} - \rho J_y\frac{\partial^2\alpha_1}{\partial t^2} - \rho J_{yz}\frac{\partial^2\alpha_2}{\partial t^2} + \int_y\int_z\left(\rho z\varphi\frac{\partial^2\vartheta}{\partial t^2} + \rho z a_1\frac{\partial^2\beta_1}{\partial t^2} + \rho z a_2\frac{\partial^2\beta_2}{\partial t^2} + \vartheta\rho z\frac{\partial^2\varphi}{\partial t^2} + \beta_1\rho z\frac{\partial^2 a_1}{\partial t^2}\right.$$
$$\left.+\beta_2\rho z\frac{\partial^2 a_2}{\partial t^2} + 2\rho z\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t} + 2\rho z\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t} + 2\rho z\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}\right)dzdy + \frac{\partial M_y}{\partial x} - Q_{13} + M_y(P_1) + M_y(q_1) = 0;$$

$$\rho S_z \frac{\partial^2 u}{\partial t^2} - \rho J_{yz} \frac{\partial^2 \alpha_1}{\partial t^2} - \rho J_z \frac{\partial^2 \alpha_2}{\partial t^2} + \int_y \int_z \left(\rho y \varphi \frac{\partial^2 \vartheta}{\partial t^2} + \rho y a_1 \frac{\partial^2 \beta_1}{\partial t^2} + \rho y a_2 \frac{\partial^2 \beta_2}{\partial t^2} + \vartheta \rho y \frac{\partial^2 \varphi}{\partial t^2} + \beta_1 \rho y \frac{\partial^2 a_1}{\partial t^2} + \right.$$

$$\left. + \beta_2 \rho y \frac{\partial^2 a_2}{\partial t^2} + 2\rho y \frac{\partial \vartheta}{\partial t}\frac{\partial \varphi}{\partial t} + 2\rho y \frac{\partial \beta_1}{\partial t}\frac{\partial a_1}{\partial t} + 2\rho y \frac{\partial \beta_2}{\partial t}\frac{\partial a_2}{\partial t} \right) dzdy + \frac{\partial M_z}{\partial x} - Q_{12} + M_z(P_1) + M_z(q_1) = 0;$$

$$\rho S_y \frac{\partial^2 v}{\partial t^2} - \rho S_z \frac{\partial^2 w}{\partial t^2} + \rho J_\rho \frac{\partial^2 \theta}{\partial t^2} + \frac{\partial M_x}{\partial x} + M_x(P_2, P_3) + M_x(q_2, q_3) = 0. \quad (5.2)$$

Natural boundary conditions are:

$$\left[-N_x + N_x(\varphi_1)\right]\delta u\big|_x = 0; \left[M_y - M_y(\varphi_1)\right]\delta\alpha_1\big|_x = 0; \left[M_z - M_z(\varphi_1)\right]\delta\alpha_2\big|_x = 0;$$

$$\left[-Q_{12} + Q_{12}(\varphi_2)\right]\delta v\big|_x = 0; \left[-Q_{13} + Q_{13}(\varphi_3)\right]\delta w\big|_x = 0; \left[-M_x + M_x(\varphi_2, \varphi_3)\right]\delta\theta\big|_x = 0. \quad (5.3)$$

Natural initial conditions are:

$$\left[\rho F \frac{\partial u}{\partial t} - \rho S_y \frac{\partial \alpha_1}{\partial t} - \rho S_z \frac{\partial \alpha_2}{\partial t} + \int_y \int_z \left(\rho \varphi \frac{\partial \vartheta}{\partial t} + \rho a_1 \frac{\partial \beta_1}{\partial t} + \rho a_2 \frac{\partial \beta_2}{\partial t} + \right.\right.$$

$$\left.\left. + \rho \vartheta \frac{\partial \varphi}{\partial t} + \rho \beta_1 \frac{\partial a_1}{\partial t} + \rho \beta_2 \frac{\partial a_2}{\partial t} \right) dzdy \right] \delta u \Big|_t = 0;$$

$$\left[\rho F \frac{\partial v}{\partial t} + \rho S_y \frac{\partial \theta}{\partial t}\right]\delta v\Big|_t = 0; \quad \left[\rho F \frac{\partial w}{\partial t} + \rho S_z \frac{\partial \theta}{\partial t}\right]\delta w\Big|_t = 0;$$

$$\left[-\rho S_y \frac{\partial u}{\partial t} - \rho J_y \frac{\partial \alpha_1}{\partial t} - \rho J_{zy} \frac{\partial \alpha_2}{\partial t} - \int_y \int_z \left(\rho z \varphi \frac{\partial \vartheta}{\partial t} + \rho z a_1 \frac{\partial \beta_1}{\partial t} + \rho z a_2 \frac{\partial \beta_2}{\partial t} + \right.\right.$$

$$\left. + \rho z \vartheta \frac{\partial \varphi}{\partial t} + \rho z \beta_1 \frac{\partial a_1}{\partial t} + \rho z \beta_2 \frac{\partial a_2}{\partial t} dzdy \right] \delta\alpha_1 \Big|_t = 0;$$

$$\left[-\rho S_z \frac{\partial u}{\partial t} - \rho J_{yz} \frac{\partial \alpha_1}{\partial t} - \rho J_z \frac{\partial \alpha_2}{\partial t} - \int_y \int_z \left(\rho y \varphi \frac{\partial \vartheta}{\partial t} + \rho y a_1 \frac{\partial \beta_1}{\partial t} + \rho y a_2 \frac{\partial \beta_2}{\partial t} + \right.\right.$$

$$\left.\left. + \rho y \vartheta \frac{\partial \varphi}{\partial t} + \rho y \beta_1 \frac{\partial a_1}{\partial t} + \rho y \beta_2 \frac{\partial a_2}{\partial t} \right) dzdy \right] \delta\alpha_2 \Big|_t = 0; \quad \left[\rho S_z \frac{\partial w}{\partial t} - \rho S_y \frac{\partial v}{\partial t} + \rho J_p \frac{\partial \theta}{\partial t}\right]\delta\theta \Big|_t = 0. \quad (5.4)$$

The system of differential equations of the vibrations with account of torsional and transverse-shear functions has the form:

$$-\rho\varphi \frac{\partial^2 u}{\partial t^2} + \rho\varphi z \frac{\partial^2 \alpha_1}{\partial t^2} + \rho\varphi z \frac{\partial^2 \alpha_2}{\partial t^2} - \rho\varphi^2 \frac{\partial^2 \vartheta}{\partial t^2} - \rho\varphi a_1 \frac{\partial^2 \beta_1}{\partial t^2} - \rho\varphi a_2 \frac{\partial^2 \beta_2}{\partial t^2} -$$

$$-\vartheta\rho\varphi \frac{\partial^2 \varphi}{\partial t^2} - \beta_1 \rho\varphi \frac{\partial^2 a_1}{\partial t^2} - \beta_2 \rho\varphi \frac{\partial^2 a_2}{\partial t^2} - 2\rho\varphi \frac{\partial \vartheta}{\partial t}\frac{\partial \varphi}{\partial t} - 2\rho\varphi \frac{\partial \beta_1}{\partial t}\frac{\partial a_1}{\partial t} - 2\rho\varphi \frac{\partial \beta_2}{\partial t}\frac{\partial a_2}{\partial t} +$$

$$+ \frac{\partial}{\partial x}(\varphi\sigma_{11}) - \frac{\partial \varphi}{\partial x}\sigma_{11} - \frac{\partial \varphi}{\partial y}\sigma_{12} - \frac{\partial \varphi}{\partial z}\sigma_{13} + \varphi P_1 + \varphi \bar{q}_1 = 0;$$

$$-\rho a_1\frac{\partial^2 u}{\partial t^2}+\rho z a_1\frac{\partial^2\alpha_1}{\partial t^2}+\rho y a_1\frac{\partial^2\alpha_2}{\partial t^2}-\rho a_1\varphi\frac{\partial^2\vartheta}{\partial t^2}-\rho a_1{}^2\frac{\partial^2\beta_1}{\partial t^2}-\rho a_1 a_2\frac{\partial^2\beta_2}{\partial t^2}-\vartheta\rho a_1\frac{\partial^2\varphi}{\partial t^2}-$$
$$-\beta_1\rho a_1\frac{\partial^2 a_1}{\partial t^2}-\beta_2\rho a_1\frac{\partial^2 a_2}{\partial t^2}-2\rho a_1\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}-2\rho a_1\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}-2\rho a_1\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}+$$
$$+\frac{\partial}{\partial x}(a_1\sigma_{11})-\frac{\partial a_1}{\partial x}\sigma_{11}-\frac{\partial a_1}{\partial y}\sigma_{12}-\frac{\partial a_1}{\partial z}\sigma_{13}+a_1P_1+a_1\bar{q}_1=0;$$
$$-\rho a_2\frac{\partial^2 u}{\partial t^2}+\rho a_2{}^2\frac{\partial^2\alpha_1}{\partial t^2}+\rho a_2 y\frac{\partial^2\alpha_2}{\partial t^2}-\rho a_2\varphi\frac{\partial^2\vartheta}{\partial t^2}-\rho a_1 a_2\frac{\partial^2\beta_1}{\partial t^2}-\rho a_2{}^2\frac{\partial^2\beta_2}{\partial t^2}-\vartheta\rho a_2\frac{\partial^2\varphi}{\partial t^2}-$$
$$-\beta_1\rho a_2\frac{\partial^2 a_1}{\partial t^2}-\beta_2\rho a_2\frac{\partial^2 a_2}{\partial t^2}-2\rho a_2\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}-2\rho a_2\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}-2\rho a_2\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}+$$
$$+\frac{\partial}{\partial x}(a_2\sigma_{11})-\frac{\partial a_2}{\partial x}\sigma_{11}-\frac{\partial a_2}{\partial y}\sigma_{12}-\frac{\partial a_2}{\partial z}\sigma_{13}+a_2P_1+a_2\bar{q}_1=0.$$
(5.5)

Natural boundary conditions on linear twist and transverse shears are:

$$\left[(-\sigma_{11}+\varphi_1)\varphi\right]\delta\vartheta\big|_x=0;\ \left[(-\sigma_{11}+\varphi_1)a_1\right]\delta\beta_1\big|_x=0;\ \left[(-\sigma_{11}+\varphi_1)a_2\right]\delta\beta_2\big|_x=0.$$
(5.6)

Natural initial conditions on linear twist and transverse shears are:

$$\Big[\rho\varphi\frac{\partial u}{\partial t}-\rho\varphi z\frac{\partial\alpha_1}{\partial t}-\rho\varphi y\frac{\partial\alpha_2}{\partial t}+\rho\varphi^2\frac{\partial\vartheta}{\partial t}+\rho\varphi a_1\frac{\partial\beta_1}{\partial t}+$$
$$+\rho\varphi a_2\frac{\partial\beta_2}{\partial t}+\rho\varphi\vartheta\frac{\partial\varphi}{\partial t}+\rho\varphi\beta_1\frac{\partial\alpha_1}{\partial t}+\rho\varphi\beta_2\frac{\partial\alpha_2}{\partial t}\Big]\delta\vartheta\Big|_t=0;$$
$$\Big[\rho a_1\frac{\partial u}{\partial t}-\rho a_1 z\frac{\partial\alpha_1}{\partial t}-\rho a_1 y\frac{\partial\alpha_2}{\partial t}+\rho\varphi a_1\frac{\partial\vartheta}{\partial t}+\rho a_1{}^2\frac{\partial\beta_1}{\partial t}+\rho a_1 a_2\frac{\partial\beta_2}{\partial t}+$$
$$+\rho a_1\vartheta\frac{\partial\varphi}{\partial t}+\rho a_1\beta_1\frac{\partial\alpha_1}{\partial t}+\rho a_1\beta_2\frac{\partial\alpha_2}{\partial t}\Big]\delta\beta_1\Big|_t=0;$$
(5.7)
$$\Big[\rho a_2\frac{\partial u}{\partial t}-\rho a_2 z\frac{\partial\alpha_1}{\partial t}-\rho a_2 y\frac{\partial\alpha_2}{\partial t}+\rho\varphi a_2\frac{\partial\vartheta}{\partial t}+\rho a_1 a_2\frac{\partial\beta_1}{\partial t}+\rho a_2{}^2\frac{\partial\beta_2}{\partial t}+$$
$$+\rho\vartheta a_2\frac{\partial\varphi}{\partial t}+\rho a_2\beta_1\frac{\partial\alpha_1}{\partial t}+\rho a_2\beta_2\frac{\partial\alpha_2}{\partial t}\Big]\delta\beta_2\Big|_t=0.$$

The system of differential equations of the vibration of the particles of the bar in stresses has the form:

$$-\rho F\frac{\partial^2 u}{\partial t^2}+S_y\rho\frac{\partial^2\alpha_1}{\partial t^2}+S_z\rho\frac{\partial^2\alpha_2}{\partial t^2}+\int_y\int_z\Big(-\rho\varphi\frac{\partial^2\vartheta}{\partial t^2}-\rho a_1\frac{\partial^2\beta_1}{\partial t^2}-\rho a_2\frac{\partial^2\beta_2}{\partial t^2}-\rho\vartheta\frac{\partial^2\varphi}{\partial t^2}-$$
$$-\rho\beta_1\frac{\partial^2 a_1}{\partial t^2}-\rho\beta_2\frac{\partial^2 a_2}{\partial t^2}-2\rho\frac{\partial\vartheta}{\partial t}\frac{\partial\varphi}{\partial t}-2\rho\frac{\partial\beta_1}{\partial t}\frac{\partial a_1}{\partial t}-2\rho\frac{\partial\beta_2}{\partial t}\frac{\partial a_2}{\partial t}+$$
$$+\frac{\partial\sigma_{11}}{\partial x}+\frac{\partial\sigma_{12}}{\partial y}+\frac{\partial\sigma_{13}}{\partial z}+P_1+q_1\Big)dzdy\times(\vartheta\delta\varphi+\beta_1\delta a_1+\beta_2\delta a_2)=0.$$
(5.8)

Natural boundary conditions on sought for functions of torsion and transverse shears are:

$$\left[(-\sigma_{11}+\varphi_1)\vartheta\right]\delta\varphi\big|_x=0;\quad\left[(-\sigma_{11}+\varphi_1)\beta_1\right]\delta a_1\big|_x=0;\quad\left[(-\sigma_{11}+\varphi_1)\beta_2\right]\delta a_2\big|_x=0;$$

$$\left[\left(-\sigma_{12}+f_1\right)\vartheta\right]\delta\varphi\big|_y=0;\quad \left[\left(-\sigma_{12}+f_1\right)\beta_1\right]\delta a_1\big|_y=0;\quad \left[\left(-\sigma_{12}+f_1\right)\beta_2\right]\delta a_2\big|_y=0;$$
$$\left[\left(-\sigma_{13}+g_1\right)\vartheta\right]\delta\varphi\big|_z=0;\quad \left[\left(-\sigma_{13}+g_1\right)\beta_1\right]\delta a_1\big|_z=0;\quad \left[\left(-\sigma_{13}+g_1\right)\beta_2\right]\delta a_2\big|_z=0. \qquad (5.9)$$

Natural initial conditions are:

$$\left[\rho F\frac{\partial u}{\partial t}-\rho S_y\frac{\partial\alpha_1}{\partial t}-\rho S_z\frac{\partial\alpha_2}{\partial t}+\int_y\int_z\left(\rho\varphi\frac{\partial\vartheta}{\partial t}+\rho a_1\frac{\partial\beta_1}{\partial t}+\rho a_2\frac{\partial\beta_2}{\partial t}+\rho\vartheta\frac{\partial\varphi}{\partial t}+\right.\right.$$
$$\left.\left.+\rho\beta_1\frac{\partial a_1}{\partial t}+\rho\beta_2\frac{\partial a_2}{\partial t}\right)dzdy\left(\vartheta\delta\varphi+\beta_1\delta a_1+\beta_2\delta a_2\right)\right]\Bigg|_t=0. \qquad (5.10)$$

With Cauchy relation, Hooke's law for the bars is written in the following form:

$$\sigma_{11}=E\varepsilon_{11}=E\left(\frac{\partial u}{\partial x}-z\frac{\partial\alpha_1}{\partial x}-y\frac{\partial\alpha_2}{\partial x}+\varphi\frac{\partial\vartheta}{\partial x}+a_1\frac{\partial\beta_1}{\partial x}+a_2\frac{\partial\beta_2}{\partial x}+\vartheta\frac{\partial\varphi}{\partial x}+\beta_1\frac{\partial a_1}{\partial x}+\beta_2\frac{\partial a_2}{\partial x}\right);$$
$$\sigma_{12}=G\varepsilon_{12}=G\left[\frac{\partial v}{\partial x}+z\frac{\partial\theta}{\partial x}-\alpha_2+\vartheta\frac{\partial\varphi}{\partial y}+\beta_1\frac{\partial a_1}{\partial y}+\beta_2\frac{\partial a_2}{\partial y}\right];$$
$$\sigma_{13}=G\varepsilon_{13}=G\left[\frac{\partial w}{\partial x}-y\frac{\partial\theta}{\partial x}-\alpha_1+\vartheta\frac{\partial\varphi}{\partial z}+\beta_1\frac{\partial a_1}{\partial z}+\beta_2\frac{\partial a_2}{\partial z}\right]. \qquad (5.11)$$

From (5.11), internal efforts and moments are calculated as follows:

$$N_x=\int_y\int_z\sigma_{11}dzdy=E\left[F\frac{\partial u}{\partial x}-S_y\frac{\partial\alpha_1}{\partial x}-S_z\frac{\partial\alpha_2}{\partial x}+\int_y\int_z\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z a_1dzdy\frac{\partial\beta_1}{\partial x}+\right.$$
$$\left.+\int_y\int_z a_2dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z\frac{\partial a_2}{\partial x}dzdy\right];$$
$$Q_{12}=\int_y\int_z\sigma_{12}dzdy=G\left[F\frac{\partial v}{\partial x}+S_y\frac{\partial\theta}{\partial x}-F\alpha_2+\vartheta\int_y\int_z\frac{\partial\varphi}{\partial y}dzdy+\beta_1\int_y\int_z\frac{\partial a_1}{\partial y}dzdy+\beta_2\int_y\int_z\frac{\partial a_2}{\partial y}dzdy\right];$$
$$Q_{13}=\int_y\int_z\sigma_{13}dzdy=G\left[F\frac{\partial w}{\partial x}-S_z\frac{\partial\theta}{\partial x}-F\alpha_1+\vartheta\int_y\int_z\frac{\partial\varphi}{\partial z}dzdy+\beta_1\int_y\int_z\frac{\partial a_1}{\partial z}dzdy+\beta_2\int_y\int_z\frac{\partial a_2}{\partial z}dzdy\right];$$
$$M_y=\int_y\int_z z\sigma_{11}dzdy=E\left[S_y\frac{\partial u}{\partial x}-J_y\frac{\partial\alpha_1}{\partial x}-J_{yz}\frac{\partial\alpha_2}{\partial x}+\int_y\int_z z\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z za_1dzdy\frac{\partial\beta_1}{\partial x}+\right.$$
$$\left.+\int_y\int_z za_2dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z z\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z z\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z z\frac{\partial a_2}{\partial x}dzdy\right];$$
$$M_z=\int_y\int_z y\sigma_{11}dzdy=E\left[S_z\frac{\partial u}{\partial x}-J_{zy}\frac{\partial\alpha_1}{\partial x}-J_y\frac{\partial\alpha_2}{\partial x}+\int_y\int_z y\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z ya_1dzdy\frac{\partial\beta_1}{\partial x}+\right.$$
$$\left.+\int_y\int_z ya_2dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z y\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z y\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z y\frac{\partial a_2}{\partial x}dzdy\right];$$
$$M_x=\int_y\int_z\left(z\sigma_{12}-y\sigma_{13}\right)dzdy=G\left[S_z\frac{\partial v}{\partial x}-S_y\frac{\partial w}{\partial x}+J_p\frac{\partial\theta}{\partial x}+S_z\alpha_1-S_y\alpha_2+\right.$$

$$+\vartheta\int_y\int_z\left(z\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial z}\right)dzdy+\beta_1\int_y\int_z G\left(z\frac{\partial a_1}{\partial y}-y\frac{\partial a_1}{\partial z}\right)dzdy+\beta_2\int_y\int_z G\left(z\frac{\partial a_2}{\partial y}-y\frac{\partial a_2}{\partial z}\right)dzdy\Bigg];$$

$$\int_y\int_z \varphi\sigma_{11}dzdy=E\Bigg[\int_y\int_z \varphi dzdy\frac{\partial u}{\partial x}-\int_y\int_z z\varphi dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z y\varphi dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z \varphi^2 dzdy\frac{\partial\vartheta}{\partial x}+$$

$$+\int_y\int_z a_1\varphi dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z a_2\varphi dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z \varphi\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z \varphi\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z \varphi\frac{\partial a_2}{\partial x}dzdy\Bigg];$$

$$\int_y\int_z a_1\sigma_{11}dzdy=E\Bigg[\int_y\int_z a_1dzdy\frac{\partial u}{\partial x}-\int_y\int_z za_1dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z ya_1dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z \varphi a_1dzdy\frac{\partial\vartheta}{\partial x}+$$

$$+\int_y\int_z a_1^2dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z a_1a_2dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z a_1\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z a_1\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z a_1\frac{\partial a_2}{\partial x}dzdy\Bigg];$$

$$\int_y\int_z a_2\sigma_{11}dzdy=E\Bigg[\int_y\int_z a_2dzdy\frac{\partial u}{\partial x}-\int_y\int_z za_2dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z ya_2dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z \varphi a_2dzdy\frac{\partial\vartheta}{\partial x}+$$

$$+\int_y\int_z a_1a_2dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z a_2^2dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z a_2\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z a_2\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z a_2\frac{\partial a_2}{\partial x}dzdy\Bigg].$$

(5.12)

From relationship (5.12) we get the equation of state of the bar:

$$EF\frac{\partial u}{\partial x}-ES_y\frac{\partial\alpha_1}{\partial x}-ES_z\frac{\partial\alpha_2}{\partial x}+\int_y\int_z E\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Ea_1dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z Ea_2dzdy\frac{\partial\beta_2}{\partial x}+$$

$$+\vartheta\int_y\int_z E\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z E\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z E\frac{\partial a_2}{\partial x}dzdy-N_x=0;$$

$$GF\frac{\partial v}{\partial x}+GS_y\frac{\partial\theta}{\partial x}-GF\alpha_2+\vartheta\int_y\int_z G\frac{\partial\varphi}{\partial y}dzdy+\beta_1\int_y\int_z G\frac{\partial a_1}{\partial y}dzdy+\beta_2\int_y\int_z G\frac{\partial a_2}{\partial y}dzdy-Q_{12}=0;$$

$$GF\frac{\partial w}{\partial x}-GS_z\frac{\partial\theta}{\partial x}-GF\alpha_1+\vartheta\int_y\int_z G\frac{\partial\varphi}{\partial z}dzdy+\beta_1\int_y\int_z G\frac{\partial a_1}{\partial z}dzdy+\beta_2\int_y\int_z G\frac{\partial a_2}{\partial z}dzdy-Q_{13}=0;$$

$$ES_y\frac{\partial u}{\partial x}-EJ_y\frac{\partial\alpha_1}{\partial x}-EJ_{yz}\frac{\partial\alpha_2}{\partial x}+\int_y\int_z Ez\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Eza_1dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z Eza_2dzdy\frac{\partial\beta_2}{\partial x}+$$

$$+\vartheta\int_y\int_z Ez\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z Ez\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z Ez\frac{\partial a_2}{\partial x}dzdy-M_y=0;$$

$$ES_z\frac{\partial u}{\partial x}-EJ_z\frac{\partial\alpha_1}{\partial x}-EJ_{yz}\frac{\partial\alpha_2}{\partial x}+\int_y\int_z Ey\varphi dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Eya_1dzdy\frac{\partial\beta_1}{\partial x}+\int_y\int_z Eya_2dzdy\frac{\partial\beta_2}{\partial x}+$$

$$+\vartheta\int_y\int_z Ey\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z Ey\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z Ey\frac{\partial a_2}{\partial x}dzdy-M_z=0;$$

$$GS_z\frac{\partial v}{\partial x}-GS_y\frac{\partial w}{\partial x}+GJ_p\frac{\partial\theta}{\partial x}+GS_z\alpha_1-GS_y\alpha_2+\vartheta\int_y\int_z G\left(z\frac{\partial\varphi}{\partial y}-y\frac{\partial\varphi}{\partial z}\right)dzdy+$$

$$+\beta_1\int_y\int_z G\left(z\frac{\partial a_1}{\partial y}-y\frac{\partial a_1}{\partial z}\right)dzdy+\beta_2\int_y\int_z G\left(z\frac{\partial a_2}{\partial y}-y\frac{\partial a_2}{\partial z}\right)dzdy-M_x=0;$$

$$\int_y\int_z E\varphi dzdy\frac{\partial u}{\partial x}-\int_y\int_z Ez\varphi dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z Ey\varphi dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z E\varphi^2 dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Ea_1\varphi dzdy\frac{\partial\beta_1}{\partial x}+$$

$$+\int_y\int_z Ea_2\varphi dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z E\varphi\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z E\varphi\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z E\varphi\frac{\partial a_2}{\partial x}dzdy-\int_y\int_z \varphi\sigma_{11}dzdy=0;$$

$$\int_y\int_z Ea_1 dzdy\frac{\partial u}{\partial x}-\int_y\int_z Eza_1 dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z Eya_1 dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z \varphi a_1 dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Ea_1^2 dzdy\frac{\partial\beta_1}{\partial x}+$$

$$+\int_y\int_z Ea_1a_2 dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z Ea_1\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z Ea_1\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z Ea_1\frac{\partial a_2}{\partial x}dzdy-\int_y\int_z a_1\sigma_{11}dzdy=0;$$

$$\int_y\int_z Ea_2 dzdy\frac{\partial u}{\partial x}-\int_y\int_z Eza_2 dzdy\frac{\partial\alpha_1}{\partial x}-\int_y\int_z Eya_2 dzdy\frac{\partial\alpha_2}{\partial x}+\int_y\int_z E\varphi a_2 dzdy\frac{\partial\vartheta}{\partial x}+\int_y\int_z Ea_1a_2 dzdy\frac{\partial\beta_1}{\partial x}+$$

$$+\int_y\int_z Ea_2^2 dzdy\frac{\partial\beta_2}{\partial x}+\vartheta\int_y\int_z Ea_2\frac{\partial\varphi}{\partial x}dzdy+\beta_1\int_y\int_z Ea_2\frac{\partial a_1}{\partial x}dzdy+\beta_2\int_y\int_z Ea_2\frac{\partial a_2}{\partial x}dzdy-\int_y\int_z a_2\sigma_{11}dzdy=0.$$

II. CONCLUSION.

The sequence of solution of the equations, obtained from Hamilton-Ostrogradsky's variation principle:

I. The system of equations (5.2) is solved under static loading with corresponding boundary conditions (5.3). After internal efforts and moments are defined, stresses σ_{ij} in cross section of the bar are determined.

II. Under static loading the equations (5.5) with corresponding boundary conditions (5.6) are solved.

III. The equation (5.8) with corresponding boundary conditions (5.9) is solved under static loading.

IV. The system of equations of the bar is solved (5.13).

So, twelve parameters of the bar are determined under spatial static loading. Under dynamic loading these parameters are the initial data.

Vibrations of the bars under dynamic loading:

I. On the basis of initial conditions (5.4) the parameters of the bar are determined in the beginning of calculation. Further the system of equations (5.5) is solved with corresponding boundary conditions (5.6).

II. Initial conditions are determined according to formula (5.10). Further the equation (5.8) with boundary conditions (5.9) is solved.

III. The equations (5.8) with boundary conditions (5.9) are solved. Here initial conditions are formed by the formulae (5.10).

IV. The system of equations (5.13) is solved.

So, twelve parameters of the bar are determined under spatial dynamic loading of the bar.

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