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# Fibonacci Sequence and some Applications of Trigonometry Functions

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***Abstract*—The Fibonacci sequence is a series of numbers where a number is found by adding up the two numbers before it. Starting with 0 and 1, the sequence goes 0,1,1,2,3,5,8,13,21,34, and so forth. Written as a rule, the expression is xn= xn−1+ xn−2.**

**Webb & Parberry proved in 1969 a startling trigonometric identity involving Fibonacci numbers. We will Provide some approves and applications of Fibonacci sequence and trigonometric functions**

***Index Terms— Fibonacci , Fibonacci spiral***

## I. FIBONACCI SEQUENCE

In this paper, we establish a few basic results pertaining to this fairly developed area of mathematics. The Fibonacci sequence is a series of numbers where a number is found by adding up the two numbers before it. Starting with 0 and 1, the sequence goes 0,1,1,2,3,5,8,13,21,34,... and so forth.

Written as a rule, the expression is xn = xn−1 + xn−2. Named after Fibonacci, also known as Leonardo of Pisa or Leonardo Pisano, Fibonacci numbers were first introduced in his Liber abaci in 1202. The son of a Pisan merchant, Fibonacci traveled widely and traded extensively. Math was incredibly important to those in the trading industry, and his passion for numbers was cultivated in his youth

Knowledge of numbers is said to have first originated in the Hindu-Arabic arithmetic system, which Fibonacci studied while growing up in North Africa. Prior to the publication of Liber abaci, the Latin-speaking world had yet to be introduced to the decimal number system. He wrote many books about geometry, commercial arithmetic and irrational numbers. He also helped develop the concept of zero.

### *A. Fibonacci spiral*

A Fibonacci spiral is a series of connected quarter-circles drawn inside an array of squares with Fibonacci numbers for dimensions. The squares fit perfectly together because of the nature of the sequence, where the next number is equal to the sum of the two before it. Any two successive Fibonacci numbers have a ratio very close to the Golden Ratio, which is roughly 1.618034. The larger the pair of Fibonacci numbers, the closer the approximation. The spiral and resulting rectangle are known as the Golden Rectangle.[1]

Fibonacci numbers verify many identities, such as trigonometric identities See e.g[2],[3],[4],In Corollary 10 ,of [5] , For

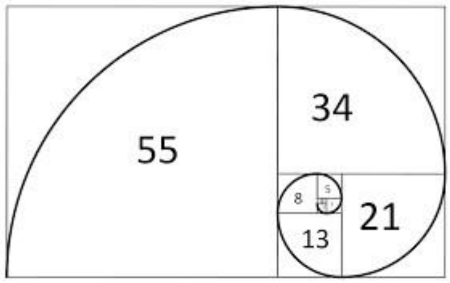


Fig. 1. The shapes of spiral looks like a Hurricane, follow the Fibonacci sequence

$n \geq 2$ n is even number

$$u_n(x,y) = x \prod_{k=1}^{(n-2)/2} (x^2 + 4y\, cos^2 \frac{k\pi}{n}) \qquad (1)$$

For $n \geq 2$ n is odd number

$$u_n(x,y) = x \prod_{k=1}^{(n-1)/2} (x^2 + 4y\, cos^2 \frac{k\pi}{n}) \qquad (2)$$

As in [6] By replacing $x$ and $y$ by 1 in equations above , we get an intriguing one which states that for $n \geq 1$

$$F_n = \prod_{k=1}^{(n-1)/2} (1 + 4cos^2 \frac{k\pi}{n}) \qquad (3)$$

This is an immediate consequence of equations (1),(2), since for $1 \leq k \leq n/2$ ,

$$cos\frac{k\pi}{n} = -cos\frac{(n-k)\pi}{n} \qquad (4)$$

It is clear from the preceding (1),(2)that there is a precise correspondence between the polynomial factors ofun(x,y) and those of un(x,1) = fn(x). Thus, it suffices to consider only those of fn(x). Also, it is clear that, except for the factor x, the only polynomial factors of fn(x) with integral coefficients contain only even powers of x. While we are not able to say in every case which even polynomials are factors of some fn(x).

### *B. The Logarithmic of Spiral*

The logarithmic spiral of figure 1 is a spiral whose polar equation is given by

$$r = ae^{b\theta}, \quad (5)$$

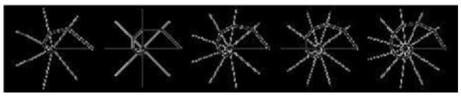

Fig. 2. golden spiral

where r is the distance from the origin, theta is the angle from the x-axis, and a and b are arbitrary constants. The logarithmic spiral is also known as the growth spiral, equiangular spiral, and spira mirabilis. It can be expressed parametrically as

$$x = r\cos\theta = a\cos\theta e^{b\theta} \quad (6)$$

$$x = r\sin\theta = a\sin\theta e^{b\theta} \quad (7)$$

This spiral is related to Fibonacci numbers, the golden ratio, and the golden rectangle, and is sometimes called the golden spiral. The logarithmic spiral can be built from similarly spaced rays by starting at a point laterally one ray, and drawing the vertical to a neighboring ray. As the number of rays approaches infinity, the sequence of segments approaches the smooth logarithmic spiral[7]

The logarithmic spiral was first studied by Descartes in 1638 and Jakob Bernoulli. Bernoulli was so fascinated by the spiral that he had one engraved on his tombstone (although the engraver did not draw it true to form) together with the words ”eadem mutata resurgo” (”I shall arise the same though changed”). Torricelli worked on it independently and found the length of the curve (MacTutor Archive).

The rate of change of radius is

$$\frac{d\theta}{dr} = abe^{b\theta} = br \quad (8)$$

and the angle between the tangent and radial line at the point $(r,\theta)$ is

$$\Omega = tan^{-1}\frac{r}{\frac{dr}{d\theta}} = cot^{-1}b \quad (9)$$

So, as ω → 0, ω → π /2 and the spiral approaches a circle.

### *C. Fibonacci Sequence and inverse of tangent function*

Let be the Fibonacci sequence given in 0,1,1,2,3,5,8,13,21,34 and so forth and using

$$tan(2x) = \frac{2tanx}{1 - tan^2x} \quad (10)$$

it is easy to verify the Leanhard Euler identity

$$tan^{-1}(1) = tan^{-1}(1/2) + tan^{-1}(1/3) = \pi/4 \quad (11)$$

The tangent inverse function contain some beginning elements of Fibonacci Sequence ,but the sequence is infinite. Hence we need a general formula which holds for any integral value of n. For this purpose we consider the following equation

$$F_{n-1}F_{n+1} - F_{n2} = (-1)_n \quad (12)$$

For any integer $n$ , The equation in (12) is called Cassinis identity. The proof of this equation follows from the principle of mathematical induction and the definition of Fibonacci sequence. For its proof readers may refer[8]. Form (12) we have

$$F_{n+1} = \frac{F_{n+2}F_n + (-1)^n}{F_{n+1}} = \frac{F_{n+2}F_n + (-1)^n}{F_{n+2} - F_n} \quad (13)$$

We can extend the (13) by putting $n = 2p$ (an even integer) in the above relation, we get

$$F_{2p+1} = \frac{F_{2p+2}F_2p + 1}{F_{2p+1}} = \frac{F_{2p+2}F_n + 1}{F_{2p+2} - F_2p} \quad (14)$$

Taking$cot^{-1}(F_{2p+1}) = \alpha$ ,$cot^{-1}(F_{2p}) \neq \beta$ , $cot^{-1}(F_{2p+2}) = \gamma$ it is easy to show that $cot(\beta-\gamma) = cot\alpha$ , $\beta = \alpha+\gamma$ which it means that $cot_{-1}(F_{2p}) = cot_{-1}(F_{2p+1}) + cot_{-1}(F_{2p+2})$ (15) For $n \geq 0$ , $n$ is odd similar argument.

## II. Fibonacci Polynomial

Consider the polynomials an(x) defined recursively by a0(x) = 0, a1(x) = x, an+1(x) = xan(x) + an−1(x). (16)

Observe that an(1) = an, the Fibonacci numbers. Recalling the standard Chebychev polynomials are related to these polynomials. Recalling the standard equation (as mentioned in the introduction) one has the following. The recursion is expressed formally by the generating function

$$\sum_{n\geq 1} a_n(x)y^n = \frac{y}{1 - xy - y^2}. \quad (17)$$

The characteristic polynomial (in terms of y) 1−xy −y2 (for each fixed x) has the roots . Note that αβ = −1. Therefore,

$$a_n(x) = \frac{\frac{1}{\alpha^{n+1}} - \frac{1}{\beta^{\ n+1}}}{\alpha \neq \beta} = (-1)^n \frac{\alpha^{n+1} \neq \beta^{\ n+1}}{\alpha \neq \beta} \quad (18)$$

Now it is clearly defined the way to find the roots of an(x) (they correspond to β/α) being a nontrivial n + 1-th root of the unit circle; we get

$$a_n(x) = \prod_{r=1}^{n-1}(x - 2icos\frac{r\pi}{n}) \quad (19)$$

By using( 3), We get

$$\begin{aligned} a_n &= a_n(1) \\ &= \textstyle\prod_{r=1}^{n-1}(1 - 2icos\frac{r\pi}{n}) \\ &= \textstyle\prod_{r=1}^{[(n-1)/2]}(1 + 4cos^2\frac{r\pi}{n}) \\ &= \textstyle\prod_{r=1}^{[(n-1)/2]}(3 + 2cos^2\frac{2r\pi}{n}) \end{aligned}$$

## III. GOLDEN RATIO

The golden ratio, given by $\lambda_1 = \frac{1+\sqrt{2}}{2}$ has been called many names throughout history. Some of these names include the golden number, golden proportion, golden mean, golden cut, golden section, divine proportion, the Fibonacci number and the mean of Phidias. Greek mathematicians defined it as the "division of a line in mean and extreme ratio" [9] Since ancient times, the golden ratio has been of interest to many people in varying disciplines. The interest in the golden ratio goes back to at least 2600 B.C. when the Egyptians were constructing the Great Pyramid. Theories suggest that the Egyptians were aware of the golden ratio and used it during the building of the Great Pyramid. Calculations show that the ratio between the base and hypotenuse of the right triangle inside the Great Pyramid is approximately 0.61762 which is very close to the reciprocal of the golden ratio. Thousands of years later in 1497, Italian mathematician Luca Pacioli wrote De Divina Proportione, which is thought to be the first book written about the golden ratio.

The golden ratio can be used to construct what are called golden rectangles, which are considered the most aestheticallypleasing rectangles. What makes these rectangles special is that the ratio of the length to the width is the golden ratio.The golden ratio has also appeared in Greek sculptures, paintings and pottery and in ancient furniture and architectural design. Famous painters Georges Seurat and Leonardo da Vinci were known to include golden rectangles and golden ratios in their paintings. Architectural structures that used the golden rectangles include the Parthenon in Athens, Greece, and the Cathedral of Chartes and Tower of Saint Jacques in Paris, France. One can also construct golden right angled triangles which √ are right triangles that have λ 1 as the hypotenuse and

λ1 and 1 as side lengths. Another fascinating fact is that certain proportions of human anatomy have shown evidence of the golden ratio and golden rectangle. Ancient Greeks noted that the human "head fits nicely into a golden rectangle". In addition, the golden ratio appears in the human hand and human bones. Because of these special relationships, we sometimes refer to the golden ratio as "the number of our physical body".[10] In addition, the golden ratio appears in the human hand and human bones. Because of these special relationships, we sometimes refer to the golden ratio as "the number of our physical body".

We have shown that the eigenvalues of matrix F are λ1 = and We have also shown that these values are used to derive the generalized Fibonacci numbers. The relationship between the generalized Fibonacci numbers and the golden ratio does not stop there. In fact, we will show that

$\frac{n+1}{g_n} \rightarrow\ _1$ as $n \to \infty$

Let $x_n = \frac{g_{n+1}}{g_n}$. Assume $g_0 \geq 0$ , $g_1 \geq 0$. Then $(x_n)$ is welldefined.

For $n \geq 1$, $x_n = \frac{g_n}{g_{n+1}} = \frac{g_n}{g_n+g_{n-1}} = \frac{1}{1+\frac{g_{n-1}}{g_n}} = \frac{1}{1+x_{n-1}}$.

Consider $x_n - x_{n+1} = \frac{1}{1+x_{n-1}} - \frac{1}{1+x_n}$.

Then , $x_n - x_{n+1} = \frac{x_n - x_{n-1}}{(1+x_{n-1})(1+x_n)}$.

Since $x_0 = \frac{g_0}{g_1} \geq 0$ and $x_1 = \frac{g_1}{g_2} = \frac{g_1}{g_0+g_1} = c > 0$, we show by induction that $x_n > 0$ for all $n \geq 1$.

Assume that $x_n > 0$ for all $k \leq n$. Then, $x_{n+1} = \frac{1}{1+x_n} > 0$.

Therefore, $x_n > 0$ for all $n \geq 1$

We conclude that, $(1 + x_{n-1})(1 + x_n) > 1 + c$ for $n \geq 2$.

Then , $|x_n - x_{n+1}| < \frac{|x_n - x_{n-1}|}{1+c}$ for all $n \geq 2$,

$|x_2 - x_1| = |\frac{g_2}{g_3} - \frac{g_1}{g_2}| = |\frac{g_0+g_1}{g_0+2g_1} - \frac{g_1}{g_0+g_1}| = k > 0$.

By using interation, we will get $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for $n \geq 2$ which we verify by mathematical induction.

Assume $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for all $k \geq n$.

Then, $|x_{n+1} - x_{n+2}| < \frac{|x_{n+1} - x_n|}{1+c} < \frac{k}{(1+c)^n}$

Therefore, $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ for $n \geq 2$

Finally, we estimate the difference between arabitrary terms $x_m$ and $x_n$. For $m < n$ ,

$|x_m - x_n| = |(x_m - x_{m+1}) + (x_{m+1} - x_{m+2}) + ... + (x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)|$

Then, $|x_m - x_n| \leq |(x_m - x_{m+1}) + (x_{m+1} - x_{m+2}) + ... + (x_{n-2} - x_{n-1}) + (x_{n-1} - x_n)|$

Using $|x_n - x_{n+1}| < \frac{k}{(1+c)^{n-1}}$ we obtain, $|x_m - x_n| \leq \frac{k}{(1+c)^{m-1}} + \frac{k}{(1+c)^m} + ... + \frac{k}{(1+c)^{n-3}} + \frac{k}{(1+c)^{n-2}}$,

$|x_m - x_n| \leq \frac{\frac{k}{(1+c)^{m-1}}}{1-\frac{1}{1+c}} = \frac{k}{c(1+c)^{m-2}}$

If $\epsilon > 0$, choose $N$ such that $\frac{k}{c(1+c)^{N-2}} < \epsilon$

Thus, $(x_n)$ is Cauchy , so $\lim_{n\to\infty} x_n = G$ exists.

Considering $x_n = \frac{1}{1+x_{n-1}}$ and taking limits we obtain,

$G = \frac{1}{1+G}$. This yields $G^2 + G - 1 = 0$ whose solutions are $\frac{-1\pm\sqrt{5}}{2}$. Since $(x_n) > 0$ for all $n \geq 1$ we $G = \frac{-1+\sqrt{5}}{2} > 0$.

Therefore for $n \geq 1$,

$\lim_{n\to\infty} \frac{n+1}{g_n} = \lim_{n\to\infty} \frac{1}{\frac{g_n}{g_{n+1}}} = \lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{G} = \frac{1+\sqrt{5}}{2} = \lambda_1$

$g$

Therefore,

$$\lim_{n\to\infty} \frac{g_{n+1}}{g_n} = \frac{1+\sqrt{5}}{2} \quad (20)$$

## IV. CONCLUSION

In this paper, we have listed only some types of trigonometry functions. There are so many results which are not studied here. The paper is focused on the some contents of papers [3],[7] ,and [6]. Fibonacci Sequence leads to very good question " Is Mathematics Invented or Discovered?".

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